A DIAGRAMMATIC CATEGORIFICATION OF THE Q-SCHUR ALGEBRA

MARCO MACKAAY, MARKO STOŠIĆ, AND PEDRO VAZ

ABSTRACT. In this paper we categorify the q-Schur algebra $\mathbf{S}_q(n,d)$ as a quotient of Khovanov and Lauda's diagrammatic 2-category $\mathcal{U}(\mathfrak{s}l_n)$ [16]. We also show that our 2-category contains Soergel's [33] monoidal category of bimodules of type A, which categorifies the Hecke algebra $H_q(d)$, as a full sub-2-category if $d \leq n$. For the latter result we use Elias and Khovanov's diagrammatic presentation of Soergel's monoidal category of type A [8].

CONTENTS

1. INTRODUCTION

There is a well-known relation, called *Schur-Weyl duality* or *reciprocity*, between the polynomial representations of homogeneous degree d of the general linear group $\mathrm{GL}(n,\mathbb{Q})$ and the finite-dimensional representations of the symmetric group on d letters S_d. Recall that all irreducible polynomial representations of $\mathrm{GL}(n,\mathbb{Q})$ of homogeneous degree d occur in the decomposition of $V^{\otimes d}$, where $V = \mathbb{Q}^n$ is the natural representation of $\mathrm{GL}(n,\mathbb{Q})$. Instead of the $\mathrm{GL}(n,\mathbb{Q})$-action, we can consider the $\mathbf{U}(\mathfrak{gl}_n)$-action, without loss of generality. A key observation for Schur-Weyl duality is that the permutation action of S_d on $V^{\otimes d}$ commutes with the action of $\mathbf{U}(\mathfrak{gl}_n)$. Furthermore, we have

$$\mathbb{Q}[S_d] \cong \mathrm{End}_{\mathbf{U}(\mathfrak{gl}_n)}(V^{\otimes d})$$

if $n \geq d$.

By definition, the *Schur algebra* is the other centralizer algebra

$$S(n,d) := \mathrm{End}_{S_d}(V^{\otimes d}).$$

It is well known that both $\mathbf{U}(\mathfrak{sl}_n)$ and $\mathbf{U}(\mathfrak{gl}_n)$ map surjectively onto $S(n,d)$, for any $d > 0$. Therefore we can also define $S(n,d)$ as the image of the map

$$\mathbf{U}(\mathfrak{gl}_n) \to \mathrm{End}_{\mathbb{Q}}(V^{\otimes d}),$$

which is the definition used in this paper. Both $S(n,d)$ and $\mathbb{Q}[S_d]$ are split semi-simple finite-dimensional algebras, and the *double centralizer property* above implies that the categories of finite-dimensional modules $S(n,d)-\text{mod}$ and $S_d-\text{mod}$ are equivalent, for $n\geq d$.

There are two more facts of interest to us. The first is that there actually exists a concrete functor which gives rise to the above mentioned equivalence. For $n\geq d$, there exists an embedding of $\mathbb{Q}[S_d]$ in $S(n,d)$, which induces the so called *Schur functor*

$$S(n,d)-\text{mod}\longrightarrow S_d-\text{mod}.$$

As it turns out, this functor is an equivalence.

The second fact of interest to us is that the Schur algebras $S(n,d)$ for various values of n and d are related. If $n\leq m$, then $S(n,d)$ can be embedded into $S(m,d)$. A more complicated relation is the following: for any $k\in\mathbb{N}$, there is a surjection

$$S(n,d+nk)\to S(n,d).$$

This surjection is compatible with the projections of $\mathbf{U}(\mathfrak{gl}_n)$ and $\mathbf{U}(\mathfrak{sl}_n)$ onto the Schur algebras. With these surjections, the Schur algebras form an inverse system. As it turns out, the projections of $\mathbf{U}(\mathfrak{sl}_n)$ onto the Schur algebras give rise to an embedding

$$\mathbf{U}(\mathfrak{sl}_n)\subset\oplus_{d=0}^{n-1}\lim_{\leftarrow k}S(n,d+nk).$$

To get a similar embedding for $\mathbf{U}(\mathfrak{gl}_n)$, one needs to consider generalized Schur algebras. We do not give the details of this generalization, because we will not need it. We refer the interested reader to [7].

All the facts recollected above have q-analogues, which involve the quantum groups $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$, the Hecke algebra $H_q(d)$, the q-Schur algebra $S_q(n,d)$, and their respective finite-dimensional representations over $\mathbb{Q}(q)$.

If one is only interested in the finite-dimensional representations of $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$, which can all be decomposed into weight spaces, it is easier to work with Lusztig's idempotented version of these quantum groups, denoted $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ and $\dot{\mathbf{U}}(\mathfrak{sl}_n)$. In these idempotented versions, the Cartan subalgebras are "replaced" by algebras generated by orthogonal idempotents corresponding to the weights. The kernel of the surjection $\dot{\mathbf{U}}(\mathfrak{gl}_n)\to S_q(n,d)$ is simply the ideal generated by all idempotents corresponding to the $\mathfrak{gl}_n$-weights which do not appear in the decomposition of $V^{\otimes d}$. The same is true for the kernel of $\dot{\mathbf{U}}(\mathfrak{sl}_n)\to S_q(n,d)$, using $\mathfrak{sl}_n$-weights. We will say more about $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ and $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ in the next section.

We are interested in the *categorification* of the q-algebras above, the relations between them and the applications to low-dimensional topology. By a categorification of a q-algebra we mean a monoidal category or a 2-category whose Grothendieck group, tensored by $\mathbb{Q}(q)$, is isomorphic to that q-algebra.

As a matter of fact, all of them have been categorified already, and some of them in more than one way. Soergel defined a category of bimodules over polynomial rings in d variables, which he proved to categorify $H_q(d)$. Elias and Khovanov gave a diagrammatic version of the Soergel category. Grojnowski and Lusztig [12] were the first to categorify $S_q(n,d)$, using categories of perverse sheaves on products of partial flag varieties. Subsequently Mazorchuk and Stroppel constructed a categorification using representation theoretic techniques [28] and so did Williamson [39] for $n=d$ using singular Soergel bimodules. Khovanov and Lauda have provided a diagrammatic

2-category $\mathcal{U}(\mathfrak{s}l_n)$ which categorifies $\dot{\mathbf{U}}(\mathfrak{sl}_n)$. Rouquier [32] followed a more representation theoretic approach to the categorification of the quantum groups. The precise relation of his work with Khovanov and Lauda's remains unclear. We note that the categorifications mentioned above have been obtained for arbitrary root data. However, this paper is only about type A and we will not consider other types.

Our interest is in the diagrammatic approach, by which $H_q(d)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ have already been categorified. The goal of this paper is to define a diagrammatic categorification of $S_q(n,d)$. Recall that the objects of $\mathcal{U}(\mathfrak{s}l_n)$ are the weights of $\mathfrak{sl}_n$, which label the regions in the diagrams which constitute the 2-morphisms. Our idea is quite simple: define a new 2-category $\mathcal{U}(\mathfrak{gl}_n)$ just as $\mathcal{U}(\mathfrak{s}l_n)$ but switch to $\mathfrak{gl}_n$-weights, which we conjecture to give a categorification of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$. Next we mod out $\mathcal{U}(\mathfrak{gl}_n)$ by all diagrams which have regions labeled by weights not appearing in the decomposition of $V^{\otimes d}$. This way we obtain a 2-category $\mathcal{S}(n,d)$ and the main result of this paper is the proof that it indeed categorifies $S_q(n,d)$.

There are two good reasons for switching to $\mathfrak{gl}_n$-weights, besides giving a conjectural categorification of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$. It is easier to say explicitly which $\mathfrak{gl}_n$-weights do not appear in $V^{\otimes d}$, as we will show in the next section. Also, while working on our paper we found a sign mistake in what Khovanov and Lauda call their signed categorification of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ [17]. Fortunately it does not affect their unsigned version, but the corrected signed version loses a nice property, the cyclicity. We discovered that with $\mathfrak{gl}_n$-weights there is a different sign convention which solves the problem, at least for $\mathcal{S}(n,d)$.

On our way of proving the main result of this paper we obtain some other interesting results:

- For $n \geq d$, we define a fully faithful 2-functor from Soergel's category of bimodules to $\mathcal{S}(n,d)$, which categorifies the well-known inclusion $H_q(d) \subset S_q(n,d)$ explained in Section 2.
- We define functors $\mathcal{S}(n,d) \to \mathcal{S}(m,d)$ when $n \leq m$. We are not (yet) able to prove that these are faithful, although we strongly suspect that they are. We know that they are not full, but suspect that they are "almost full" in a sense that we will explain in Section 7.
- We define essentially surjective full 2-functors

$$\mathcal{S}(n,d+kn) \to \mathcal{S}(n,d)$$

  which categorify the surjections above.
- We show that Khovanov and Lauda's 2-representation of $\mathcal{U}(\mathfrak{s}l_n)$ on the equivariant cohomology of flag varieties descends to $\mathcal{S}(n,d)$.
- We conjecture how to categorify the irreducible representations of $S_q(n,d)$ using $\mathcal{S}(n,d)$. Khovanov and Lauda's categorification of these representations, using the so-called cyclotomic quotients, should be equivalent to a quotient of ours.

Understanding the precise relation with the other categorifications of $S_q(n,d)$ would be very important, but is left for the future. As a matter of fact, Brundan and Stroppel have already established a link between the category $\mathcal{O}$ approach to categorification and Khovanov and Lauda's (see for example [2]), which perhaps can be used to obtain an equivalence between Mazorchuk and Stroppel's categorification of the q-Schur algebra and ours. For $n = d$, Williamson's 2-category of Soergel's singular bimodules is equivalent to Khovanov and Lauda's 2-category build out of the equivariant cohomology of partial flag varieties (of flags in $\mathbb{Q}^d$) and we expect both to be equivalent to $\mathcal{S}(d,d)$.

Besides the intrinsic interest of $\mathcal{S}(n,d)$, with its combinatorics and its link to representation theory, there is also a potential application to knot theory. First recall that there is a natural surjection of the braid group onto $H_q(d)$. The Jones-Ocneanu trace of the image of a braid in $H_q(d)$ is equal to the so called HOMFLYPT knot polynomial of the braid closure. This construction has been categorified: Rouquier defined a complex of Soergel bimodules for each braid and Khovanov discovered that its Hochschild homology categorifies the Jones-Ocneanu trace, showing that in this way one obtains a homology which is isomorphic to the Khovanov-Rozansky HOMFLYPT-homology. Using Elias and Khovanov's work, Elias and Krasner [9] worked out the diagrammatic version of Rouquier's complex. Their work still remains to be extended to include the Hochschild homology. Besides this approach, which is the one most directly related to the results in this paper, we should also mention a geometric approach due to Webster and Williamson in [37] and a representation theoretic approach due to Mazorchuk and Stroppel [29].

More generally, there is a natural homomorphism from the colored braid group, with n strands colored by natural numbers whose sum is equal to d, to $S_q(n,d)$. It is not as widely advertised as the non-colored version, but one can easily obtain it from Lusztig's formulas in Section 5.2.1 in [22] or from the second part of the paper by Murakami-Ohtsuki-Yamada [30]. One can also define a colored version of the Jones-Ocneanu trace on $S_q(n,d)$ to obtain the colored HOMFLYPT knot invariant. Naturally the question arises how to categorify the colored HOMFLYPT knot polynomial. In [5] Chuang and Rouquier defined a colored version of Rouquier's complex for a braid, using a representation theoretic approach. They proved invariance under the second braid-like Reidemeister move and conjectured invariance under the third move. In [25] we defined a complex of singular Soergel bimodules, which is equivalent to the Chuang-Rouquier complex. We conjectured that the Hochschild homology of such a complex categorifies the colored HOMFLYPT knot polynomial of the braid closure. We were only able to prove our conjecture for the colors 1 and 2, due to the complexity of the calculations for general colors. Webster and Williamson subsequently showed our conjecture to be true, using a generalization of their geometric approach [38]. Cautis, Kamnitzer and Licata [3] also studied the Chuang-Rouquier complex from a geometric point of view. By the above mentioned 2-representation of $\mathcal{S}(n,d)$ into singular Soergel bimodules, it is natural to expect that one should be able to define the Chuang-Rouquier complex in $\mathcal{S}(n,d)$, such that its 2-representation gives exactly the complex of singular Soergel bimodules which we conjectured. In a forthcoming paper we will come back to this. In the meanwhile, papers have appeared in which the colored HOMFLYPT homology has been constructed using matrix factorizations (see [40, 41, 42, 43, 44]).

The outline of this paper is as follows:

- In Section 2 we recall some results on the above mentioned q-algebras. Our choice has been highly selective in an attempt to prevent this paper from becoming too long. We have only included those results which we categorify or which we need in order to categorify. We hope that this introduction makes up for what we left out.
- In Section 3 we define the 2-categories $\mathcal{U}(\mathfrak{gl}_n)$ and $\mathcal{S}(n,d)$. As said before, the first one is just a copy of Khovanov and Lauda's definition of $\mathcal{U}(\mathfrak{sl}_n)$, but with a different set of weights and a different sign convention. The second one is a quotient of the first one.
- To understand some of the properties of $\mathcal{S}(n,d)$, we first define its 2-representation in the 2-category of bimodules over polynomial rings in Section 4. Except for the different sign convention, it is the factorization of the 2-representation of [16] through $\mathcal{S}(n,d)$. The

only new feature is our interpretation of this 2-representation in terms of the categorified MOY-calculus, which we developed in [25].

- Section 5 is devoted to comparing the structure of the 2-HOM spaces of $\mathcal{U}(\mathfrak{s}l_n)$ to those of $\mathcal{S}(n,d)$. The latter ones remain a bit of a mystery to us and we can only prove just enough about them for what we need in the rest of this paper.
- In Section 6 we define a fully faithful embedding of Soergel's categorification of $H_q(d)$ into $\mathcal{S}(n,d)$. We have not yet attributed any notation to Soergel's category in this introduction, because there are actually two slightly different versions of it and we will need both, one for $d=n$ and the other for $d<n$.
- In Section 7 we prove that $\mathcal{S}(n,d)$ indeed categorifies $S_q(n,d)$. We also conjecture how to categorify the Weyl modules of $S_q(n,d)$.

2. HECKE AND q-SCHUR ALGEBRAS

In this section we recollect some facts about the q-algebras mentioned in the introduction. For details and proofs see [6] and [27] unless other references are mentioned. We work over the field $\mathbb{Q}(q)$, where q is a formal parameter.

2.1. The quantum general and special linear algebras.

Let us first recall the quantum general and special linear algebras. The $\mathfrak{gl}_n$-weight lattice is isomorphic to $\mathbb{Z}^n$. Let $\varepsilon_i = (0,\ldots,1,\ldots,0) \in \mathbb{Z}^n$, with 1 being on the ith coordinate, and $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathbb{Z}^n$, for $i=1,\ldots,n-1$. We also define the Euclidean inner product on $\mathbb{Z}^n$ by $(\varepsilon_i,\varepsilon_j)=\delta_{i,j}$.

Definition 2.1. The *quantum general linear algebra* $\mathbf{U}_q(\mathfrak{gl}_n)$ is the associative unital $\mathbb{Q}(q)$-algebra generated by K_i, K_i^{-1}, for $1,\ldots,n$, and $E_{\pm i}$, for $i=1,\ldots,n-1$, subject to the relations

$$K_iK_j = K_jK_i \quad K_iK_i^{-1} = K_i^{-1}K_i = 1$$
$$E_iE_{-j} - E_{-j}E_i = \delta_{i,j}\frac{K_iK_{i+1}^{-1} - K_i^{-1}K_{i+1}}{q-q^{-1}}$$
$$K_iE_{\pm j} = q^{\pm(\varepsilon_i,\alpha_j)}E_{\pm j}K_i$$
$$E_{\pm i}^2E_{\pm j} - (q+q^{-1})E_{\pm i}E_{\pm j}E_{\pm i} + E_{\pm j}E_{\pm i}^2 = 0 \qquad \text{if} \quad |i-j|=1$$
$$E_{\pm i}E_{\pm j} - E_{\pm j}E_{\pm i} = 0 \qquad \text{else.}$$

Definition 2.2. The *quantum special linear algebra* $\mathbf{U}_q(\mathfrak{sl}_n) \subseteq \mathbf{U}_q(\mathfrak{gl}_n)$ is the unital $\mathbb{Q}(q)$-subalgebra generated by $K_iK_{i+1}^{-1}$ and $E_{\pm i}$, for $i=1,\ldots,n-1$.

Recall that the $\mathbf{U}_q(\mathfrak{sl}_n)$-weight lattice is isomorphic to $\mathbb{Z}^{n-1}$. Suppose that V is a $\mathbf{U}_q(\mathfrak{gl}_n)$-weight representation with weights $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n$, i.e.

$$V \cong \bigoplus_\lambda V_\lambda$$

and K_i acts as multiplication by q^{λ_i} on V_λ. Then V is also a $\mathbf{U}_q(\mathfrak{sl}_n)$-weight representation with weights $\overline{\lambda} = (\overline{\lambda}_1,\ldots,\overline{\lambda}_{n-1}) \in \mathbb{Z}^{n-1}$ such that $\overline{\lambda}_j = \lambda_j - \lambda_{j+1}$ for $j=1,\ldots,n-1$. Conversely, given a $\mathbf{U}_q(\mathfrak{sl}_n)$-weight representation with weights $\mu = (\mu_1,\ldots,\mu_{n-1})$, there is not a unique choice of $\mathbf{U}_q(\mathfrak{gl}_n)$-action on V. We can fix this by choosing the action of $K_1\cdots K_n$. In terms of

weights, this corresponds to the observation that, for any $d \in \mathbb{Z}$ the equations

$$\lambda_i - \lambda_{i+1} = \mu_i \tag{2.1}$$

$$\sum_{i=1}^{n} \lambda_i = d \tag{2.2}$$

determine $\lambda = (\lambda_1, \ldots, \lambda_n)$ uniquely, if there exists a solution to (2.1) and (2.2) at all. To fix notation, we define the map $\varphi_{n,d} \colon \mathbb{Z}^{n-1} \to \mathbb{Z}^n \cup \{*\}$ by

$$\varphi_{n,d}(\mu) = \lambda$$

if (2.1) and (2.2) have a solution, and put $\varphi_{n,d}(\mu) = *$ otherwise.

Recall that $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ are both Hopf algebras, which implies that the tensor product of two of their representations is a representation again.

Both $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$ have plenty of non-weight representations, but we are not interested in them. Therefore we can restrict our attention to the Beilinson-Lusztig-MacPherson [1] idempotented version of these quantum groups, denoted $\dot{\mathrm{U}}(\mathfrak{gl}_n)$ and $\dot{\mathrm{U}}(\mathfrak{sl}_n)$ respectively. To understand their definition, recall that K_i acts as q^{λ_i} on the λ-weight space of any weight representation. For each $\lambda \in \mathbb{Z}^n$ adjoin an idempotent 1_λ to $\mathbf{U}_q(\mathfrak{gl}_n)$ and add the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda,\nu} 1_\lambda \\ E_{\pm i} 1_\lambda &= 1_{\lambda \pm \alpha_i} E_{\pm i} \\ K_i 1_\lambda &= q^{\lambda_i} 1_\lambda . \end{aligned}$$

Definition 2.3. The idempotented quantum general linear algebra is defined by

$$\dot{\mathrm{U}}(\mathfrak{gl}_n) = \bigoplus_{\lambda,\mu \in \mathbb{Z}^n} 1_\lambda \mathbf{U}_q(\mathfrak{gl}_n) 1_\mu .$$

For $\underline{\boldsymbol{i}} = (\alpha_1 i_1, \ldots, \alpha_{n-1} i_{n-1})$, with $\alpha_j = \pm$, define

$$E_{\underline{\boldsymbol{i}}} := E_{\alpha_1 i_1} \cdots E_{\alpha_{n-1} i_{n-1}}$$

and define $\underline{\boldsymbol{i}}_\Lambda \in \mathbb{Z}^n$ to be the n-tuple such that

$$E_{\underline{\boldsymbol{i}}} 1_\mu = 1_{\mu + \underline{\boldsymbol{i}}_\Lambda} E_{\underline{\boldsymbol{i}}} .$$

Similarly for $\mathbf{U}_q(\mathfrak{sl}_n)$, adjoin an idempotent 1_μ for each $\mu \in \mathbb{Z}^{n-1}$ and add the relations

$$\begin{aligned} 1_\mu 1_\nu &= \delta_{\mu,\nu} 1_\lambda \\ E_{\pm i} 1_\mu &= 1_{\mu \pm \overline{\alpha}_i} E_{\pm i} \\ K_i K_{i+1}^{-1} 1_\mu &= q^{\mu_i} 1_\mu . \end{aligned}$$

Definition 2.4. The idempotented quantum special linear algebra is defined by

$$\dot{\mathrm{U}}(\mathfrak{sl}_n) = \bigoplus_{\mu,\nu \in \mathbb{Z}^{n-1}} 1_\mu \mathbf{U}_q(\mathfrak{sl}_n) 1_\nu .$$

Note that $\dot{\mathrm{U}}(\mathfrak{gl}_n)$ and $\dot{\mathrm{U}}(\mathfrak{sl}_n)$ are both non-unital algebras, because their units would have to be equal to the infinite sum of all their idempotents. Furthermore, the only $\mathbf{U}_q(\mathfrak{gl}_n)$ and $\mathbf{U}_q(\mathfrak{sl}_n)$-representations which factor through $\dot{\mathrm{U}}(\mathfrak{gl}_n)$ and $\dot{\mathrm{U}}(\mathfrak{sl}_n)$, respectively, are the weight representations. Finally, note that there is no embedding of $\dot{\mathrm{U}}(\mathfrak{sl}_n)$ into $\dot{\mathrm{U}}(\mathfrak{gl}_n)$, because there is no embedding of the $\mathfrak{sl}_n$-weights into the $\mathfrak{gl}_n$-weights.

2.2. **The q-Schur algebra.** Let $d \in \mathbb{N}$ and let V be the natural n-dimensional representation of $\mathbf{U}_q(\mathfrak{gl}_n)$. Define

$$\Lambda(n,d) = \{\lambda \in \mathbb{N}^n \colon \ \sum_{i=1}^{n} \lambda_i = d\}$$

$$\Lambda^+(n,d) = \{\lambda \in \Lambda(n,d) \colon d \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}.$$

Recall that the weights in $V^{\otimes d}$ are precisely the elements of $\Lambda(n,d)$, and that the highest weights are the elements of $\Lambda^+(n,d)$. The highest weights correspond exactly to the irreducibles V_λ that show up in the decomposition of $V^{\otimes d}$.

As explained in the introduction, we can define the q-Schur algebra as follows:

Definition 2.5. The q-Schur algebra $S_q(n,d)$ is the image of the representation $\psi_{n,d} \colon \mathbf{U}_q(\mathfrak{gl}_n) \to \mathrm{End}_{\mathbb{Q}}(V^{\otimes d})$.

For each $\lambda \in \Lambda^+(n,d)$, the $\mathbf{U}_q(\mathfrak{gl}_n)$-action on V_λ factors through the projection $\psi_{n,d} \colon \mathbf{U}_q(\mathfrak{gl}_n) \to S_q(n,d)$. This way we obtain all irreducible representations of $S_q(n,d)$. Note that this also implies that all representations of $S_q(n,d)$ have a weight decomposition. As a matter of fact, it is well known that

$$S_q(n,d) \cong \prod_{\lambda \in \Lambda^+(n,d)} \mathrm{End}_{\mathbb{Q}}(V_\lambda).$$

Therefore $S_q(n,d)$ is a finite-dimensional split semi-simple unital algebra and its dimension is equal to

$$\sum_{\lambda \in \Lambda^+(n,d)} \dim(V_\lambda)^2 = \binom{n^2+d-1}{d}.$$

Since $V^{\otimes d}$ is a weight representation, $\psi_{n,d}$ gives rise to a homomorphism $\dot{\mathbf{U}}(\mathfrak{gl}_n) \to S_q(n,d)$, for which we use the same notation. This map is still surjective and Doty and Giaquinto, in Theorem 2.4 of [7], showed that the kernel of $\psi_{n,d}$ is equal to the ideal generated by all idempotents 1_λ such that $\lambda \notin \Lambda(n,d)$. Let $\dot{\mathbf{S}}(n,d)$ be the quotient of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$ by the kernel of $\psi_{n,d}$. Clearly we have $\dot{\mathbf{S}}(n,d) \cong S_q(n,d)$. By the above observations, we see that $\dot{\mathbf{S}}(n,d)$ has a Serre presentation. As a matter of fact, by Corollary 4.3.2 in [4], this presentation is simpler than that of $\dot{\mathbf{U}}(\mathfrak{gl}_n)$: one does not need to impose the last two Serre relations, involving cubical terms, because they are implied by the other relations and the finite dimensionality.[1]

Lemma 2.6. $\dot{\mathbf{S}}(n,d)$ is isomorphic to the associative unital $\mathbb{Q}(q)$-algebra generated by 1_λ, for $\lambda \in \Lambda(n,d)$, and $E_{\pm i}$, for $i = 1, \ldots, n-1$, subject to the relations

$$1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda$$

$$\sum_{\lambda \in \Lambda(n,d)} 1_\lambda = 1$$

$$E_{\pm i} 1_\lambda = 1_{\lambda \pm \alpha_i} E_{\pm i}$$

$$E_i E_{-j} - E_{-j} E_i = \delta_{ij} \sum_{\lambda \in \Lambda(n,d)} [\overline{\lambda}_i] 1_\lambda.$$

[1]We thank Raphaël Rouquier for pointing this out to us and giving us the reference.

We use the convention that $1_\mu X 1_\lambda = 0$, if μ or λ is not contained in $\Lambda(n,d)$. Recall that $[a]$ is the q-integer $(q^a - q^{-a})/(q - q^{-1})$.

Although there is no embedding of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ into $\dot{\mathbf{U}}(\mathfrak{gl}_n)$, the projection

$$\psi_{n,d}\colon \mathbf{U}_q(\mathfrak{gl}_n) \to S_q(n,d)$$

can be restricted to $\mathbf{U}_q(\mathfrak{sl}_n)$ and is still surjective. This gives rise to the surjection

$$\psi_{n,d}\colon \dot{\mathbf{U}}(\mathfrak{sl}_n) \to \dot{\mathbf{S}}(n,d),$$

defined by

$$\psi_{n,d}(E_{\pm i}1_\lambda) = E_{\pm i}1_{\varphi_{n,d}(\lambda)}, \tag{2.3}$$

where $\varphi_{n,d}$ was defined below equations (2.1) and (2.2). By convention we put $1_* = 0$.

As mentioned in the introduction, the q-Schur algebras for various values of n and d are related. Let $m \geq n$ and d be arbitrary. There is an obvious embedding of the set of $\mathbf{U}_q(\mathfrak{gl}_n)$-weights into the set of $\mathbf{U}_q(\mathfrak{gl}_m)$-weights, given by

$$(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0).$$

For fixed d, this gives an inclusion $\Lambda(n,d) \subseteq \Lambda(m,d)$, which we can use to define

$$\xi_{n,m} = \sum_{\lambda \in \Lambda(n,d)} 1_\lambda \in \dot{\mathbf{S}}(m,d).$$

Note that $\xi_{n,m} \neq 1$ unless $n = m$.

Definition 2.7. There is a well-defined homomorphism

$$\iota_{n,m}\colon \dot{\mathbf{S}}(n,d) \to \xi_{n,m}\dot{\mathbf{S}}(m,d)\xi_{n,m}$$

given by

$$E_{\pm i} \mapsto \xi_{n,m}E_{\pm i}\xi_{n,m} \qquad \text{and} \qquad 1_\lambda \mapsto \xi_{n,m}1_\lambda\xi_{n,m} = 1_\lambda.$$

It is easy to see that this is an isomorphism.

Definition 2.8. Suppose $d' = d + nk$, for a certain $k \in \mathbb{N}$. Then we define a homomorphism

$$\pi_{d',d}\colon \dot{\mathbf{S}}(n,d') \to \dot{\mathbf{S}}(n,d)$$

by

$$1_\lambda \mapsto 1_{\lambda - (k^n)} \quad \text{and} \quad E_{\pm i} \mapsto E_{\pm i}.$$

It is easy to check that $\pi_{d',d}$ is well-defined and surjective. It is also easy to see that

$$\pi_{d',d}\psi_{n,d'} = \psi_{n,d}$$

and that $\pi_{d',d}$ induces a linear isomorphism

$$V_\lambda \to V_{\lambda-(k^n)},$$

which intertwines the $\dot{\mathbf{S}}(n,d')$ and $\dot{\mathbf{S}}(n,d)$ actions, if $\lambda - (k^n) \in \Lambda^+(n,d)$. Of course V_λ and $V_{\lambda-(k^n)}$ are isomorphic as $\mathbf{U}_q(\mathfrak{sl}_n)$ representations. Furthermore, note that for any $d = 0, \ldots, n-1$ the set

$$(S_q(n, d+nk), \pi_{d+nk,d})_{k\in\mathbb{N}} \tag{2.4}$$

forms an inverse system, so we can form the inverse limit algebra

$$\varprojlim_k S_q(n, d+nk).$$

The following lemma is perhaps a bit surprising.

Lemma 2.9. The map $\sum_d \prod_k \psi_{n,d+nk}$, with $d = 0, \ldots, n-1$ and $k \in \mathbb{N}$, gives an embedding

$$\mathbf{U}_q(\mathfrak{sl}_n) \subset \bigoplus_{d=0}^{n-1} \lim_{\longleftarrow k} S_q(n, d+nk).$$

We also have

$$\dot{\mathbf{U}}(\mathfrak{sl}_n) \subset \bigoplus_{d=0}^{n-1} \lim_{\longleftarrow k} S_q(n, d+nk). \tag{2.5}$$

The reader should remember this embedding when reading Corollary 5.2. The results in this paragraph were taken from [1].

We need to recall two more facts about q-Schur algebras and their representations. The first is that the irreducibles V_λ, for $\lambda \in \Lambda^+(n,d)$, can be constructed as subquotients of $\dot{\mathbf{S}}(n,d)$, called Weyl modules. Let $<$ denote the lexicographic order on $\Lambda(n,d)$.

Lemma 2.10. For any $\lambda \in \Lambda^+(n,d)$, we have

$$V_\lambda \cong \dot{\mathbf{S}}(n,d)1_\lambda/[\mu > \lambda].$$

Here $[\mu > \lambda]$ is the ideal generated by all elements of the form $1_\mu x 1_\lambda$, for some $x \in \dot{\mathbf{S}}(n,d)$ and $\mu > \lambda$.

Finally, we recall a well known anti-involution on $\dot{\mathbf{S}}(n,d)$, which we will need in this paper.

Definition 2.11. We define an algebra anti-involution

$$\tau \colon \dot{\mathbf{S}}(n,d) \to \dot{\mathbf{S}}(n,d)^{\mathrm{op}}$$

by

$$\tau(1_\lambda) = 1_\lambda, \quad \tau(1_{\lambda+\alpha_i} E_i 1_\lambda) = q^{-1-\overline{\lambda}_i} 1_\lambda E_{-i} 1_{\lambda+\alpha_i}, \quad \tau(1_\lambda E_{-i} 1_{\lambda+\alpha_i}) = q^{1+\overline{\lambda}_i} 1_{\lambda+\alpha_i} E_i 1_\lambda.$$

Note that up to a shift t', we have

$$1_\mu E_{s_1} E_{s_2} \cdots E_{s_{m-1}} E_{s_m} 1_\lambda q^t \mapsto 1_\lambda E_{-s_m} E_{-s_{m-1}} \cdots E_{-s_2} E_{-s_1} 1_\mu q^{-t+t'}.$$

Our τ is the analogue of the one in [16].

2.3. **The Hecke algebra.** Recall that $H_q(n)$ is a q-deformation of the group algebra of the symmetric group on n letters.

Definition 2.12. The Hecke algebra $H_q(n)$ is the unital associative $\mathbb{Q}(q)$-algebra generated by the elements T_i, $i = 1, \ldots, n-1$, subject to the relations

$$\begin{aligned} T_i^2 &= (q^2-1)T_i + q^2 \\ T_iT_j &= T_jT_i \qquad \text{if} \quad |i-j| > 1 \\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}. \end{aligned}$$

Note that some people write q where we write q^2 and use $v = q^{-1}$ in their presentation of the Hecke algebra. It is also not uncommon to find t instead of our q.

For $q = 1$ we recover the presentation of $\mathbb{Q}[S_n]$ in terms of the simple transpositions σ_i. For any element $\sigma \in S_n$ we can define $T_\sigma = T_{i_1} \cdots T_{i_k}$, choosing a reduced expression $\sigma = \sigma_{i_1} \cdots \sigma_{i_k}$. The relations above guarantee that all reduced expressions of σ give the same element T_σ. The T_σ, for $\sigma \in S_n$, form a linear basis of $H_q(n)$.

There is a simple change of generators, which is convenient for categorification purposes. Write $b_i = q^{-1}(T_i + 1)$. Then the relations above become

$$\begin{aligned} b_i^2 &= (q + q^{-1}) b_i \\ b_i b_j &= b_j b_i \qquad \text{if} \quad |i-j| > 1 \\ b_i b_{i+1} b_i + b_{i+1} &= b_{i+1} b_i b_{i+1} + b_i. \end{aligned}$$

These generators are the simplest elements of the so called *Kazhdan-Lusztig basis*. Although the change of generators is simple, the whole change of linear bases is very complicated.

As mentioned in the introduction, there is a q-version of Schur-Weyl duality. There is a q-permutation action of $H_q(d)$ on $V^{\otimes d}$, which is induced by the R-matrix of $\mathbf{U}_q(\mathfrak{gl}_n)$ or $\mathbf{U}_q(\mathfrak{sl}_n)$ and commutes with the actions of these quantum enveloping algebras. With respect to these actions, $H_q(d)$ and $\dot{\mathbf{S}}(n,d)$ have the double centralizer property. Furthermore, their respective categories of finite-dimensional representations are equivalent.

Suppose $n \geq d$. We explicitly recall the embedding of $H_q(d)$ into $\dot{\mathbf{S}}(n,d)$. Let $1_d = 1_{(1^d)}$. Note that the $\mathbf{U}_q(\mathfrak{gl}_n)$-weight (1^d) gives the zero $\mathbf{U}_q(\mathfrak{sl}_n)$-weight, for $n = d$, and a fundamental $\mathbf{U}_q(\mathfrak{sl}_n)$-weight for $n > d$. We define the following map

$$\sigma_{n,d} \colon H_q(d) \to 1_d \dot{\mathbf{S}}(n,d) 1_d$$

by

$$\sigma_{n,d}(b_i) = 1_d E_{-i} E_i 1_d = 1_d E_i E_{-i} 1_d,$$

for $i = 1, \ldots, d-1$. It is easy to check that $\sigma_{n,d}$ is well-defined. It turns out that $\sigma_{n,d}$ is actually an isomorphism, which induces the *q-Schur functor* $\dot{\mathbf{S}}(n,d) - \mathrm{mod} \to H_q(d) - \mathrm{mod}$, where mod denotes the category of finite-dimensional modules. This functor is an equivalence. Let us state explicitly an easy implication of this equivalence, which we need in the sequel.

Lemma 2.13. Let $0 < d \leq n$ and let A be a unital associative $\mathbb{Q}(q)$-algebra. Suppose $\pi \colon \dot{\mathbf{S}}(n,d) \to A$ is a surjection of $\mathbb{Q}(q)$-algebras, such that $\pi \circ \sigma_{n,d} \colon H_q(d) \to A$ is an embedding. Then $A \cong \dot{\mathbf{S}}(n,d)$.

Proof. Recall that

$$\dot{\mathbf{S}}(n,d) \cong \prod_{\lambda \in \Lambda^+(n,d)} \mathrm{End}_{\mathbb{Q}(q)}(V_\lambda).$$

The fact that the q-Schur functor is an equivalence means that the projection of $\sigma_{n,d}(H_q(d))$ onto $\mathrm{End}_{\mathbb{Q}(q)}(V_\lambda)$ is non-zero, for any $\lambda \in \Lambda^+(n,d)$. Since all $\mathrm{End}_{\mathbb{Q}(q)}(V_\lambda)$ are simple algebras, A has to be isomorphic to the product

$$\prod_{\lambda \in \Lambda'} \mathrm{End}_{\mathbb{Q}(q)}(V_\lambda),$$

for a certain subset $\Lambda' \subseteq \Lambda^+(n,d)$. But $\pi \circ \sigma_{n,d}$ is an embedding, so $\Lambda' = \Lambda^+(n,d)$ has to hold. $\square$

3. The 2-categories $\mathcal{U}(\mathfrak{gl}_n)$ and $\mathcal{S}(n,d)$

In this section we define two 2-categories, $\mathcal{U}(\mathfrak{gl}_n)$ and $\mathcal{S}(n,d)$, using a graphical calculus analogous to Khovanov and Lauda's in [16]. We thank Khovanov and Lauda for letting us copy their definition of $\mathcal{U}_{\rightarrow}(\mathfrak{sl}_n)$. Taking their definition, we first introduce a change of weights to obtain $\mathcal{U}(\mathfrak{gl}_n)$. Then we divide by an ideal to obtain $\mathcal{S}(n,d)$.

As remarked in the introduction, our signs are slightly different from those in [16]. Khovanov and Lauda [17] corrected their sign convention in $\mathcal{U}_{\rightarrow}(\mathfrak{sl}_n)$. As it turns out, the corrected $\mathcal{U}_{\rightarrow}(\mathfrak{sl}_n)$ is no longer cyclic, which makes working with that sign convention awkward. Fortunately Khovanov and Lauda's non-signed version, $\mathcal{U}(\mathfrak{s}l_n)$, is still correct and cyclic and is isomorphic to the corrected $\mathcal{U}_{\rightarrow}(\mathfrak{sl}_n)$ [16, 17]. However, the sign convention in $\mathcal{U}(\mathfrak{s}l_n)$ is not so practical for the 2-representation into bimodules, so we have decided to stick to our own sign convention in this paper. To get from our signs back to Khovanov and Lauda's (corrected) signs in $\mathcal{U}_{\rightarrow}(\mathfrak{sl}_n)$, apply the 2-isomorphism which is the identity on all objects, 1- and 2-morphisms except the left cups and caps, on which it is given by

(3.1)  $_{i,\lambda} \mapsto (-1)^{\lambda_{i+1}+1}$  $_{i,\lambda}$ and  $_{i,\lambda} \mapsto (-1)^{\lambda_{i+1}}$  $_{i,\lambda}$.

The various parts of our definition of $\mathcal{U}(\mathfrak{gl}_n)$ and $\mathcal{S}(n,d)$ below have exactly the same order as the corresponding parts of Khovanov and Lauda's definition of $\mathcal{U}_{\rightarrow}(\mathfrak{sl}_n)$, so the reader can compare them in detail. From now on we will always write $\mathcal{U}(\mathfrak{s}l_n)$, instead of $\mathcal{U}_{\rightarrow}(\mathfrak{sl}_n)$, for the corrected signed categorification of $\dot{\mathrm{U}}(\mathfrak{sl}_n)$. Since we will never work with the unsigned version, there should be no confusion.

3.1. **The 2-category $\mathcal{U}(\mathfrak{gl}_n)$.** As already remarked in the introduction, the idea underlying the definition of $\mathcal{U}(\mathfrak{gl}_n)$ is very simple: it is obtained from $\mathcal{U}(\mathfrak{s}l_n)$ by passing from $\mathfrak{sl}_n$-weights to $\mathfrak{gl}_n$-weights.

From now on let $n \in \mathbb{N}_{>1}$ be arbitrary but fixed and let $I = \{1, 2, \ldots, n-1\}$. In the sequel we use *signed sequences* $\underline{\boldsymbol{i}} = (\alpha_1 i_1, \ldots, \alpha_m i_m)$, for any $m \in \mathbb{N}$, $\alpha_j \in \{\pm 1\}$ and $i_j \in I$. The set of signed sequences we denote SSeq. For $\underline{\boldsymbol{i}} = (\alpha_1 i_1, \ldots, \alpha_m i_m) \in \mathrm{SSeq}$ we define $\underline{\boldsymbol{i}}_\Lambda := \alpha_1 (i_1)_\Lambda + \cdots + \alpha_m (i_m)_\Lambda$, where

$$(i_j)_\Lambda = (0, 0, \ldots, 1, -1, 0 \ldots, 0),$$

such that the vector starts with $i_j - 1$ and ends with $k - 1 - i_j$ zeros. To understand these definitions, the reader should recall our definition of $E_{\underline{\boldsymbol{i}}}$ and $\underline{\boldsymbol{i}}_\Lambda$ below Definition 2.3. We also define the symmetric $\mathbb{Z}$-valued bilinear form on $\mathbb{Q}[I]$ by $i \cdot i = 2$, $i \cdot (i+1) = -1$ and $i \cdot j = 0$, for $|i - j| > 1$. Recall that $\overline{\lambda}_i = \lambda_i - \lambda_{i+1}$.

Definition 3.1. $\mathcal{U}(\mathfrak{gl}_n)$ is an additive $\mathbb{Q}$-linear 2-category. The 2-category $\mathcal{U}(\mathfrak{gl}_n)$ consists of

- objects: $\lambda \in \mathbb{Z}^n$.

The hom-category $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ between two objects λ, λ' is an additive $\mathbb{Q}$-linear category consisting of:

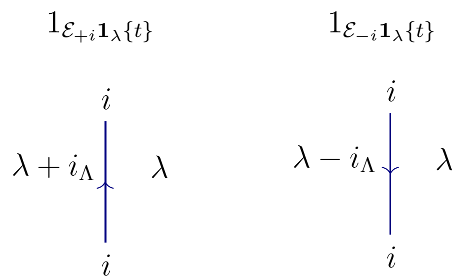
- objects[2] of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$: a 1-morphism in $\mathcal{U}(\mathfrak{gl}_n)$ from λ to λ' is a formal finite direct sum of 1-morphisms

$$\mathcal{E}_{\underline{\boldsymbol{i}}} \mathbf{1}_\lambda \{t\} = \mathbf{1}_{\lambda'} \mathcal{E}_{\underline{\boldsymbol{i}}} \mathbf{1}_\lambda \{t\} := \mathcal{E}_{\alpha_1 i_1} \cdots \mathcal{E}_{\alpha_m i_m} \mathbf{1}_\lambda \{t\}$$

[2]We refer to objects of the category $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ as 1-morphisms of $\mathcal{U}(\mathfrak{gl}_n)$. Likewise, the morphisms of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ are called 2-morphisms in $\mathcal{U}(\mathfrak{gl}_n)$.

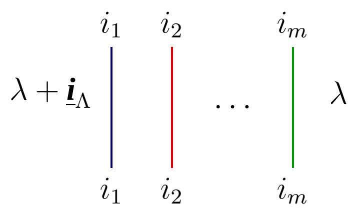
for any $t \in \mathbb{Z}$ and signed sequence $\underline{\boldsymbol{i}} \in \mathrm{SSeq}$ such that $\lambda' = \lambda + \underline{\boldsymbol{i}}_\Lambda$ and $\lambda, \lambda' \in \mathbb{Z}^n$.

- morphisms of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$: for 1-morphisms $\mathcal{E}_{\underline{\boldsymbol{i}}}\mathbf{1}_\lambda\{t\}$ and $\mathcal{E}_{\underline{\boldsymbol{j}}}\mathbf{1}_\lambda\{t'\}$ in $\mathcal{U}(\mathfrak{gl}_n)$, the hom sets $\mathcal{U}(\mathfrak{gl}_n)(\mathcal{E}_{\underline{\boldsymbol{i}}}\mathbf{1}_\lambda\{t\}, \mathcal{E}_{\underline{\boldsymbol{j}}}\mathbf{1}_\lambda\{t'\})$ of $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda')$ are graded $\mathbb{Q}$-vector spaces given by linear combinations of degree $t - t'$ diagrams, modulo certain relations, built from composites of:
  i) Degree zero identity 2-morphisms 1_x for each 1-morphism x in $\mathcal{U}(\mathfrak{gl}_n)$; the identity 2-morphisms $1_{\mathcal{E}_{+i}\mathbf{1}_\lambda}\{t\}$ and $1_{\mathcal{E}_{-i}\mathbf{1}_\lambda}\{t\}$, for $i \in I$, are represented graphically by

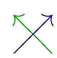
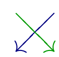
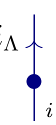
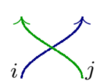
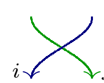


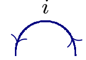
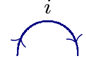
deg 0 deg 0

for any $\lambda + i_\Lambda \in \mathbb{Z}^n$ and any $\lambda - i_\Lambda \in \mathbb{Z}^n$, respectively.
More generally, for a signed sequence $\underline{\boldsymbol{i}} = (\alpha_1 i_1, \alpha_2 i_2, \dots \alpha_m i_m)$, the identity $1_{\mathcal{E}_{\underline{\boldsymbol{i}}}\mathbf{1}_\lambda\{t\}}$ 2-morphism is represented as



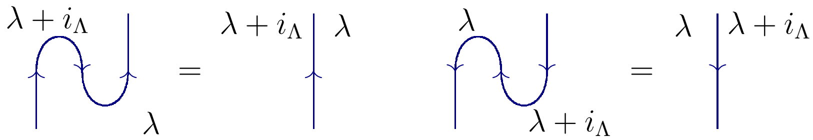
where the strand labeled i_k is oriented up if $\alpha_k = +$ and oriented down if $\alpha_k = -$. We will often place labels with no sign on the side of a strand and omit the labels at the top and bottom. The signs can be recovered from the orientations on the strands.
  ii) For each $\lambda \in \mathbb{Z}^n$ the 2-morphisms

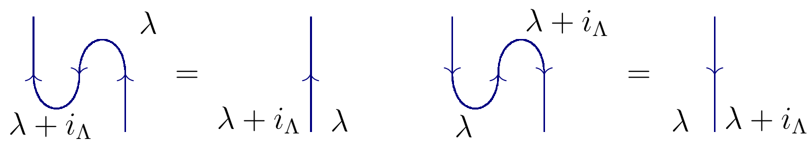
Notation:	 i,λ	 i,λ	 i,j,λ	 i,j,λ
2-morphism:	$\lambda + i_\Lambda$  λ, i	λ  $\lambda + i_\Lambda$, i	i  j λ	i  j λ
Degree:	$i \cdot i$	$i \cdot i$	$-i \cdot j$	$-i \cdot j$

Notation:	 i,λ	 i,λ	 i,λ	 i,λ
2-morphism:	 i λ	 i λ	i  λ	i  λ
Degree:	$1 + \overline{\lambda}_i$	$1 - \overline{\lambda}_i$	$1 + \overline{\lambda}_i$	$1 - \overline{\lambda}_i$

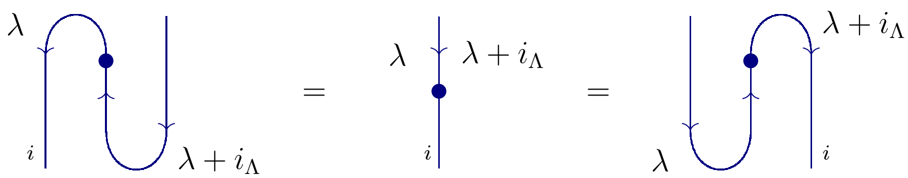
- Biadjointness and cyclicity:

  i) $\mathbf{1}_{\lambda+i_\Lambda}\mathcal{E}_{+i}\mathbf{1}_\lambda$ and $\mathbf{1}_\lambda\mathcal{E}_{-i}\mathbf{1}_{\lambda+i_\Lambda}$ are biadjoint, up to grading shifts:

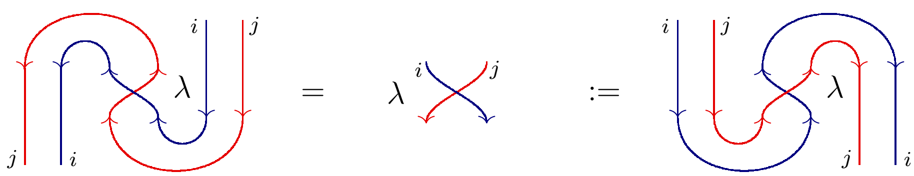
(3.2) 

(3.3) 

  ii)

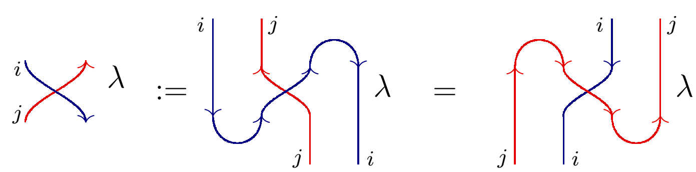
(3.4) 

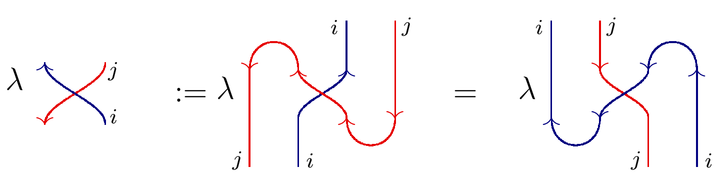
  iii) All 2-morphisms are cyclic with respect to the above biadjoint structure.[3] This is ensured by the relations (3.4), and the relations

(3.5) 

Note that we can take either the first or the last diagram above as the definition of the up-side-down crossing. We have chosen the last one above, because it is the one which matches Khovanov and Lauda's signs. The cyclic condition on 2-morphisms expressed by (3.4) and (3.5) ensures that diagrams related by isotopy represent the same 2-morphism in $\mathcal{U}(\mathfrak{gl}_n)$.

It will be convenient to introduce degree zero 2-morphisms:

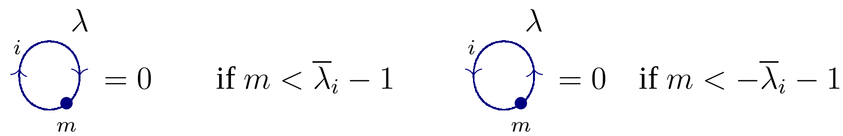
(3.6) 

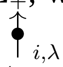
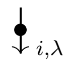
(3.7) 

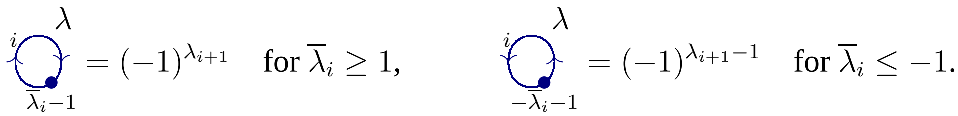
where the second equality in (3.6) and (3.7) follow from (3.5). Again we have indicated which choice of twists we use to define the sideways crossings, which is exactly the choice which matches Khovanov and Lauda's sign conventions.

[3] See [20] and the references therein for the definition of a cyclic 2-morphism with respect to a biadjoint structure.

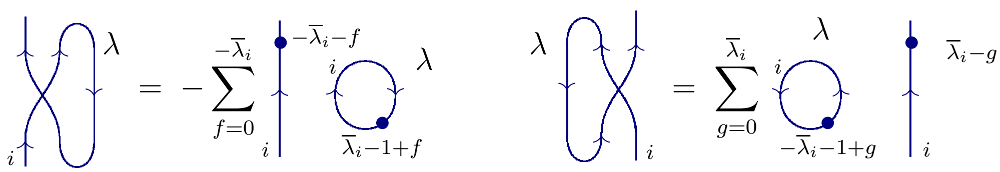
iv) All dotted bubbles of negative degree are zero. That is,

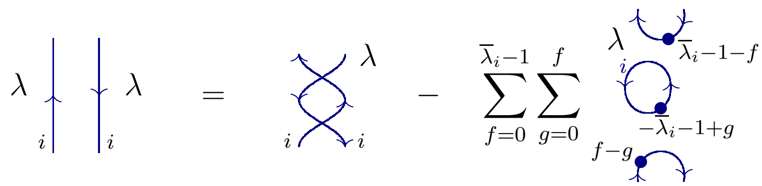
(3.8) 

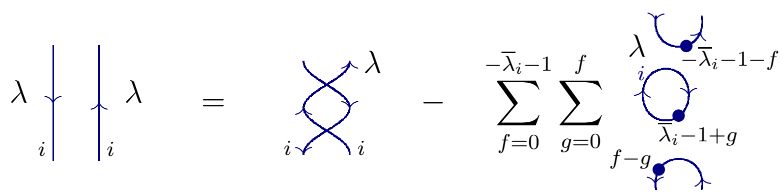
for all $m \in \mathbb{Z}_+$, where a dot carrying a label m denotes the m-fold iterated vertical composite of  or  depending on the orientation. A dotted bubble of degree zero equals ± 1:

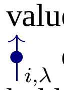
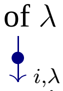
(3.9) 

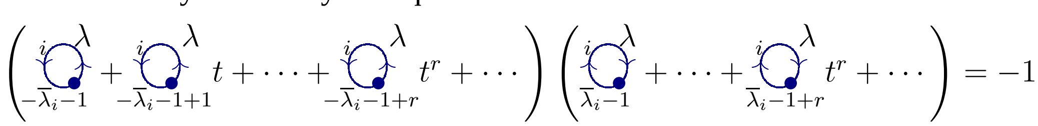
v) For the following relations we employ the convention that all summations are increasing, so that a summation of the form $\sum_{f=0}^{m}$ is zero if $m < 0$.

(3.10) 

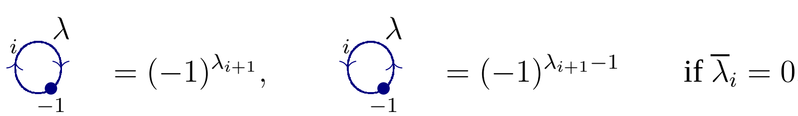
(3.11) 

(3.12) 

for all $\lambda \in \mathbb{Z}^n$ (see (3.6) and (3.7) for the definition of sideways crossings). Notice that for some values of λ the dotted bubbles appearing above have negative labels. A composite of  or  with itself a negative number of times does not make sense. These dotted bubbles with negative labels, called *fake bubbles*, are formal symbols inductively defined by the equation

(3.13) 

and the additional condition

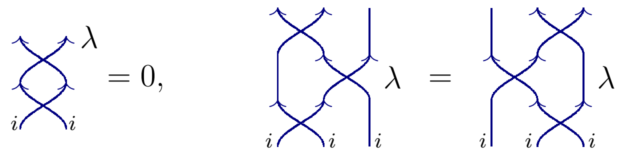


Although the labels are negative for fake bubbles, one can check that the overall degree of each fake bubble is still positive, so that these fake bubbles do not violate the

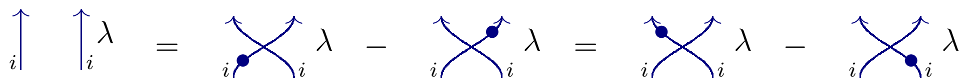
positivity of dotted bubble axiom. The above equation, called the infinite Grassmannian relation, remains valid even in high degree when most of the bubbles involved are not fake bubbles. See [20] for more details.

vi) NilHecke relations:

(3.14)



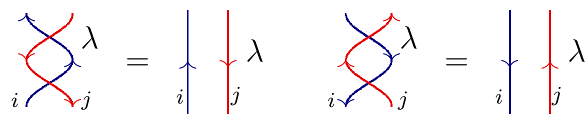
(3.15)



We will also include (3.5) for $i = j$ as an $\mathfrak{sl}_2$-relation.

- For $i \neq j$

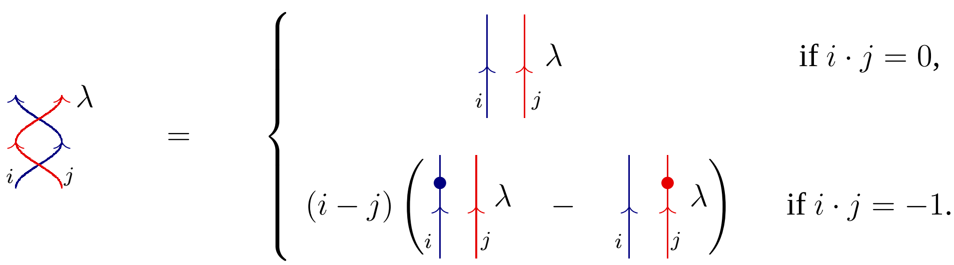
(3.16)



- The analogue of the $R(\nu)$-relations:

i) For $i \neq j$

(3.17)

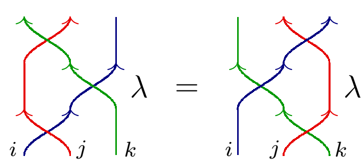


Notice that $(i - j)$ is just a sign, which takes into account the standard orientation of the Dynkin diagram.

(3.18)

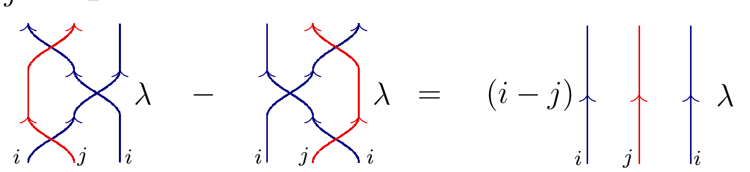
ii) Unless $i = k$ and $i \cdot j = -1$

(3.19)



For $i \cdot j = -1$

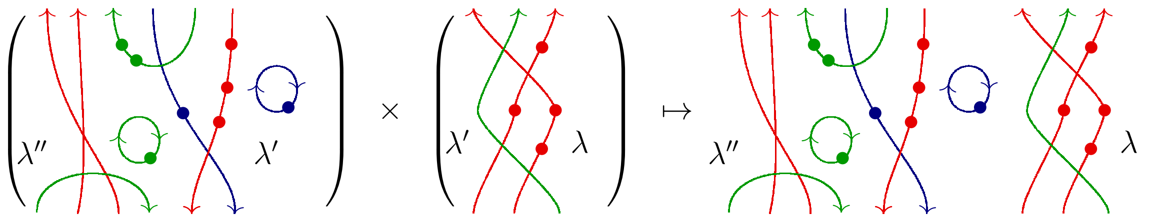
(3.20)



- The additive $\mathbb{Z}$-linear composition functor $\mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda') \times \mathcal{U}(\mathfrak{gl}_n)(\lambda', \lambda'') \to \mathcal{U}(\mathfrak{gl}_n)(\lambda, \lambda'')$ is given on 1-morphisms of $\mathcal{U}(\mathfrak{gl}_n)$ by

$$\mathcal{E}_{\underline{j}}\mathbf{1}_{\lambda'}\{t'\} \times \mathcal{E}_{\underline{i}}\mathbf{1}_{\lambda}\{t\} \mapsto \mathcal{E}_{\underline{ji}}\mathbf{1}_{\lambda}\{t+t'\} \tag{3.21}$$

for $\underline{\boldsymbol{i}}_\Lambda = \lambda - \lambda'$, and on 2-morphisms of $\mathcal{U}(\mathfrak{gl}_n)$ by juxtaposition of diagrams

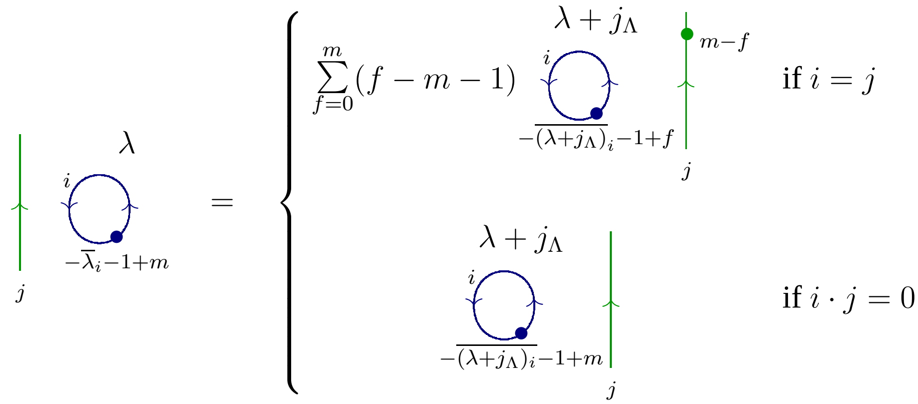


This concludes the definition of $\mathcal{U}(\mathfrak{gl}_n)$. In the next subsection we will show some further relations, which are easy consequences of the ones above.

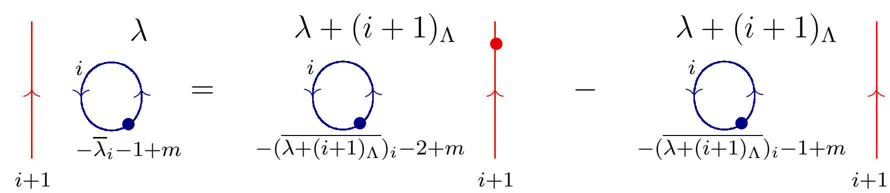
3.1.1. *Further relations in* $\mathcal{U}(\mathfrak{gl}_n)$. The following $\mathcal{U}(\mathfrak{gl}_n)$-relations follow from the relations in Definition 3.1 and are going to be used in the sequel.

Bubble slides:

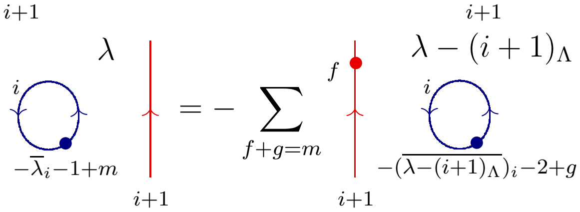
(3.22)



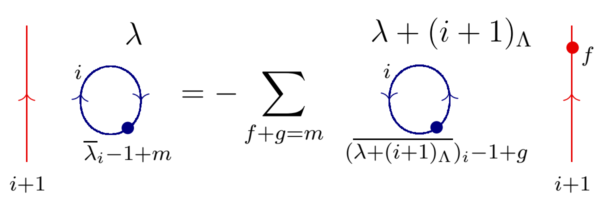
(3.23)

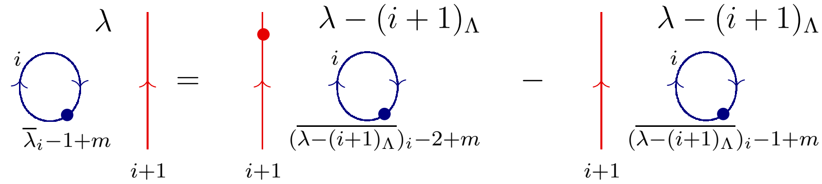


(3.24)



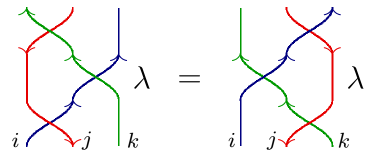
(3.25)



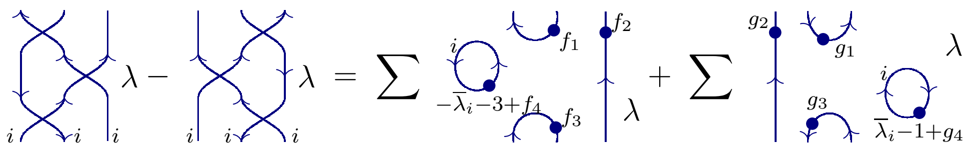
(3.26) 

If we switch labels i and $i+1$, then the r.h.s. of the above equations gets a minus sign. Bubble slides with the vertical strand oriented downwards can easily be obtained from the ones above by rotating the diagrams 180 degrees.

More Reidemeister 3 like relations. Unless $i = k = j$ we have

(3.27) 

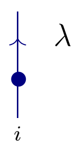
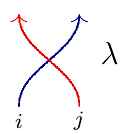
and when $i = j = k$ we have

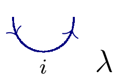
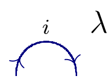
(3.28) 

where the first sum is over all $f_1, f_2, f_3, f_4 \geq 0$ with $f_1 + f_2 + f_3 + f_4 = \overline{\lambda}_i$ and the second sum is over all $g_1, g_2, g_3, g_4 \geq 0$ with $g_1 + g_2 + g_3 + g_4 = \overline{\lambda}_i - 2$. Note that the first summation is zero if $\overline{\lambda}_i < 0$ and the second is zero when $\overline{\lambda}_i < 2$.

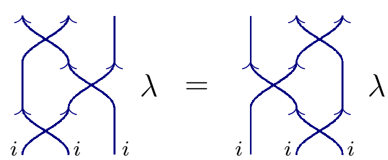
Reidemeister 3 like relations for all other orientations are determined from (3.19), (3.20), and the above relations using duality.

3.1.2. *Enriched* Hom *spaces.* For any shift t, there are 2-morphisms

 $: \mathcal{E}_{+i}\mathbf{1}_\lambda\{t\} \Rightarrow \mathcal{E}_{+i}\mathbf{1}_\lambda\{t-2\}$  $: \mathcal{E}_{+i+j}\mathbf{1}_\lambda\{t\} \Rightarrow \mathcal{E}_{+j+i}\mathbf{1}_\lambda\{t - i\cdot j\}$

 $: \mathbf{1}_\lambda\{t\} \Rightarrow \mathcal{E}_{-i+i}\mathbf{1}_\lambda\{t-(1+\overline{\lambda}_i)\}$  $: \mathcal{E}_{-i+i}\mathbf{1}_\lambda\{t\} \Rightarrow \mathbf{1}_\lambda\{t-(1-\overline{\lambda}_i)\}$

in $\mathcal{U}(\mathfrak{gl}_n)$, and the diagrammatic relation



gives rise to relations in $\mathcal{U}(\mathfrak{gl}_n)\big(\mathcal{E}_{iii}\mathbf{1}_\lambda\{t\}, \mathcal{E}_{iii}\mathbf{1}_\lambda\{t+3i\cdot i\}\big)$ for all $t \in \mathbb{Z}$.

Note that for two 1-morphisms x and y in $\mathcal{U}(\mathfrak{gl}_n)$ the 2hom-space $\mathrm{Hom}_{\mathcal{U}(\mathfrak{gl}_n)}(x,y)$ only contains 2-morphisms of degree zero and is therefore finite-dimensional. Following Khovanov and Lauda we introduce the graded 2hom-space

$$\mathrm{HOM}_{\mathcal{U}(\mathfrak{gl}_n)}(x,y) = \oplus_{t\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{U}(\mathfrak{gl}_n)}(x\{t\}, y),$$

which is infinite-dimensional. We also define the 2-category $\mathcal{U}(\mathfrak{gl}_n)^*$ which has the same objects and 1-morphisms as $\mathcal{U}(\mathfrak{gl}_n)$, but for two 1-morphisms x and y the vector space of 2-morphisms is defined by

$$\mathcal{U}(\mathfrak{gl}_n)^*(x,y) = \mathrm{HOM}_{\mathcal{U}(\mathfrak{gl}_n)}(x,y). \tag{3.29}$$

3.2. **The 2-category $\mathcal{S}(n,d)$.** Fix $d \in \mathbb{N}_{>0}$. As explained in Section 2, the q-Schur algebra $\dot{\mathbf{S}}(n,d)$ can be seen as a quotient of $\mathcal{U}(\mathfrak{gl}_n)$ by the ideal generated by all idempotents corresponding to the weights that do not belong to $\Lambda(n,d)$. It is then natural to define the 2-category $\mathcal{S}(n,d)$ as a quotient of $\mathcal{U}(\mathfrak{gl}_n)$ as follows.

Definition 3.2. The 2-category $\mathcal{S}(n,d)$ is the quotient of $\mathcal{U}(\mathfrak{gl}_n)$ by the ideal generated by all 2-morphisms containing a region with a label not in $\Lambda(n,d)$.

We remark that we only put real bubbles, whose interior has a label outside $\Lambda(n,d)$, equal to zero. To see what happens to a fake bubble, one first has to write it in terms of real bubbles with the opposite orientation using the infinite Grassmannian relation (3.13).

4. A 2-REPRESENTATION OF $\mathcal{S}(n,d)$

In this section we define a 2-functor

$$\mathcal{F}_{Bim} : \mathcal{S}(n,d)^* \to \mathbf{Bim}^*,$$

where $\mathbf{Bim}$ is the graded 2-category of bimodules over polynomial rings with rational coefficients. Recall that in the previous section (formula (3.29)), we have defined the * version of a graded 2-category, as the 2-category with the same objects and 1-morphisms, while the 2-morphisms between two 1-morphisms can have arbitrary degree.

In [16] Khovanov and Lauda defined a 2-functor Γ_d^G from $\mathcal{U}(\mathfrak{s}l_n)$ to a 2-category equivalent to a sub-2-category of $\mathbf{Bim}^*$. As one can easily verify, Γ_d^G kills any diagram with labels outside $\Lambda(n,d)$, so it descends to $\mathcal{S}(n,d)$. In this section we have rewritten this 2-functor, which we denote $\mathcal{F}_{Bim}$, in terms of categorified MOY-diagrams, because we think it might help some people to understand its definition more easily. For further comments see Section 4.3.

4.1. **Categorified MOY diagrams.** Before proceeding with the definition of $\mathcal{F}_{Bim}$, we first specify our notation for MOY diagrams and their categorification.

A colored MOY diagram [30], is an oriented trivalent graph whose edges are labeled by natural numbers (this label is also called the *color* or the *thickness* of the corresponding edge). At each trivalent vertex we have at least one incoming and one outgoing edge, and we require that at each vertex the sum of the labels of the incoming edges is equal to the sum of the labels of the outgoing edges. Moreover, in this paper we assume that all edges in MOY diagrams are oriented upwards.

To obtain a bimodule corresponding to a given colored MOY diagram, we proceed in the following way: To each edge labeled a, we associate a variables, say $\underline{x} = (x_1, \ldots, x_a)$, and to different edges we associate different variables. At every vertex (like the ones in Figure 1), we impose the relations

$$e_i(z_1,\ldots,z_{a+b}) = e_i(x_1,\ldots,x_a,y_1,\ldots,y_b)$$
$$e_i(z'_1,\ldots,z'_{a+b}) = e_i(x'_1,\ldots,x'_a,y'_1,\ldots,y'_b)$$

for all $i \in \{1,\dots,a+b\}$, where e_i is the ith elementary symmetric polynomial. In other words, at every vertex we require that an arbitrary symmetric polynomial in the variables corresponding to the incoming edges, is equal to the same symmetric polynomial in the variables corresponding to the outgoing edges.

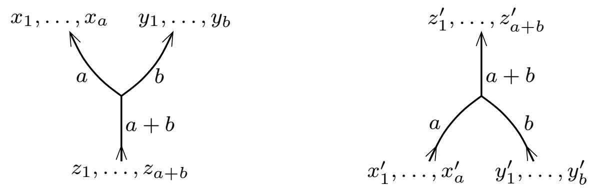
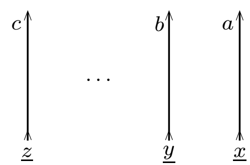


FIGURE 1. trivalent vertices

Now, to an arbitrary diagram Γ, we associate the ring R_Γ of polynomials over $\mathbb{Q}$ which are symmetric in the variables on each strand separately, modded out by the relations corresponding to all trivalent vertices.

In particular, to a graph without trivalent vertices (just strands):



we associate the ring of partially symmetric polynomials $\mathbb{Q}[\underline{x},\underline{y},\dots,\underline{z}]^{S_a\times S_b\times\cdots\times S_c}$.

In this way, the ring R_Γ associated to a MOY diagram Γ, is a bimodule over the rings of partially symmetric polynomials associated to the top (right action) and bottom end (left action) strands, respectively (remember that we are assuming that all MOY diagrams are oriented upwards, so they have a top and a bottom end). Bimodules are graded by setting the degree of any variable equal to 2.

In the rest of the paper, we will often identify the MOY diagram and the corresponding bimodule. Also, by abuse of notation, we shall call the elements of the bimodule R_Γ polynomials.

There is another way to describe these bimodules associated to MOY diagrams (see e.g. [13, 25, 39]). Fix the polynomial ring $R := \mathbb{Q}[x_1,\dots,x_d]$. For any $(a_1,\dots,a_n) \in \Lambda(n,d)$, let $R^{a_1,\dots,a_n}$ be the sub-ring of polynomials which are invariant under $S_{a_1}\times\cdots\times S_{a_n}$. To the first diagram in Figure 1 one associates the $R^{a+b}-R^{a,b}$-bimodule

$$\mathrm{Res}^{R^{a+b}}_{R^{a,b}} R^{a,b},$$

where one simply restricts the left action on $R^{a,b}$ to $R^{a+b}\subseteq R^{a,b}$. To the second diagram in Figure 1 one associates the $R^{a,b}-R^{a+b}$-bimodule

$$\mathrm{Ind}^{R^{a,b}}_{R^{a+b}} R^{a+b} := R^{a,b}\otimes_{R^{a+b}} R^{a+b}.$$

In this way, to every MOY-diagram Γ one associates a tensor product of bimodules, which is isomorphic to the bimodule R_Γ that we described in the paragraph above.

In this paper we always use R_Γ, since it is computationally easier to use polynomials than to use tensor products of polynomials.

4.2. **Definition of $\mathcal{F}_{Bim}$.** Now we can proceed with the definition of $\mathcal{F}_{Bim}\colon \mathcal{S}(n,d)^* \to \mathbf{Bim}^*$.

Let $z_1, \dots, z_d$ be variables. For convenience we shall use Khovanov and Lauda's notation $k_i = \lambda_1 + \cdots + \lambda_i$, for $i = 1, \dots, n$.

On objects $\lambda \in \Lambda(n,d)$, the 2-functor $\mathcal{F}_{Bim}$ is given by:

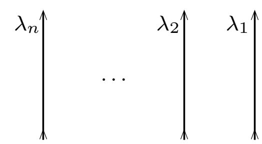
$$\lambda = (\lambda_1, \cdots, \lambda_n) \mapsto \mathbb{Q}[z_1, \dots, z_d]^{S_{\lambda_1} \times \cdots \times S_{\lambda_n}}.$$

On 1-morphisms we define $\mathcal{F}_{Bim}$ as follows:

$$1_\lambda\{t\} \mapsto \mathbb{Q}[z_1, \dots, z_d]^{S_{\lambda_1} \times \cdots \times S_{\lambda_n}}\{t\}.$$

In terms of MOY diagrams this is presented by:

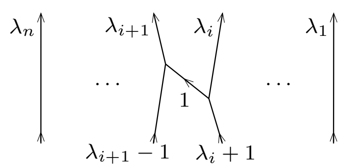
$1_\lambda\{t\} \quad \longmapsto$



Note that we are drawing the entries of λ from right to left, which is compatible with Khovanov and Lauda's convention.

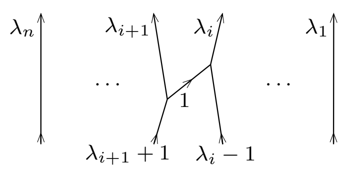
The remaining generating 1-morphisms are mapped as follows:

$\mathcal{E}_{+i}\mathbf{1}_\lambda\{t\} \quad \mapsto$



$\{t + 1 + k_{i-1} + k_i - k_{i+1}\}$

$\mathcal{E}_{-i}\mathbf{1}_\lambda\{t\} \quad \mapsto$



$\{t + 1 - k_i\}$

In both cases, the partition corresponding to the bottom strands is $\lambda + j_\Lambda$ (with j being $+i$ or $-i$). Thus, the condition we imposed on $\mathcal{S}(n,d)$ that all regions have labels from $\Lambda(n,d)$ (i.e. no region can have labels with negative entries), ensures that on the RHS above we really have MOY diagrams.

The composite $\mathcal{F}_{Bim}(\mathcal{E}_i 1_{\lambda+j_\Lambda} \mathcal{E}_j 1_\lambda)$ is given by stacking the MOY diagram corresponding to $\mathcal{E}_j 1_\lambda$ on top of the one corresponding to $\mathcal{E}_i 1_{\lambda+j_\Lambda}$. The shifts add under composition.

To define $\mathcal{F}_{Bim}$ on 2-morphisms, we give the image of the generating 2-morphisms. In the definitions the divided difference operator ∂_{xy} is used. For $p \in Q[x, y, \dots]$ it is given by

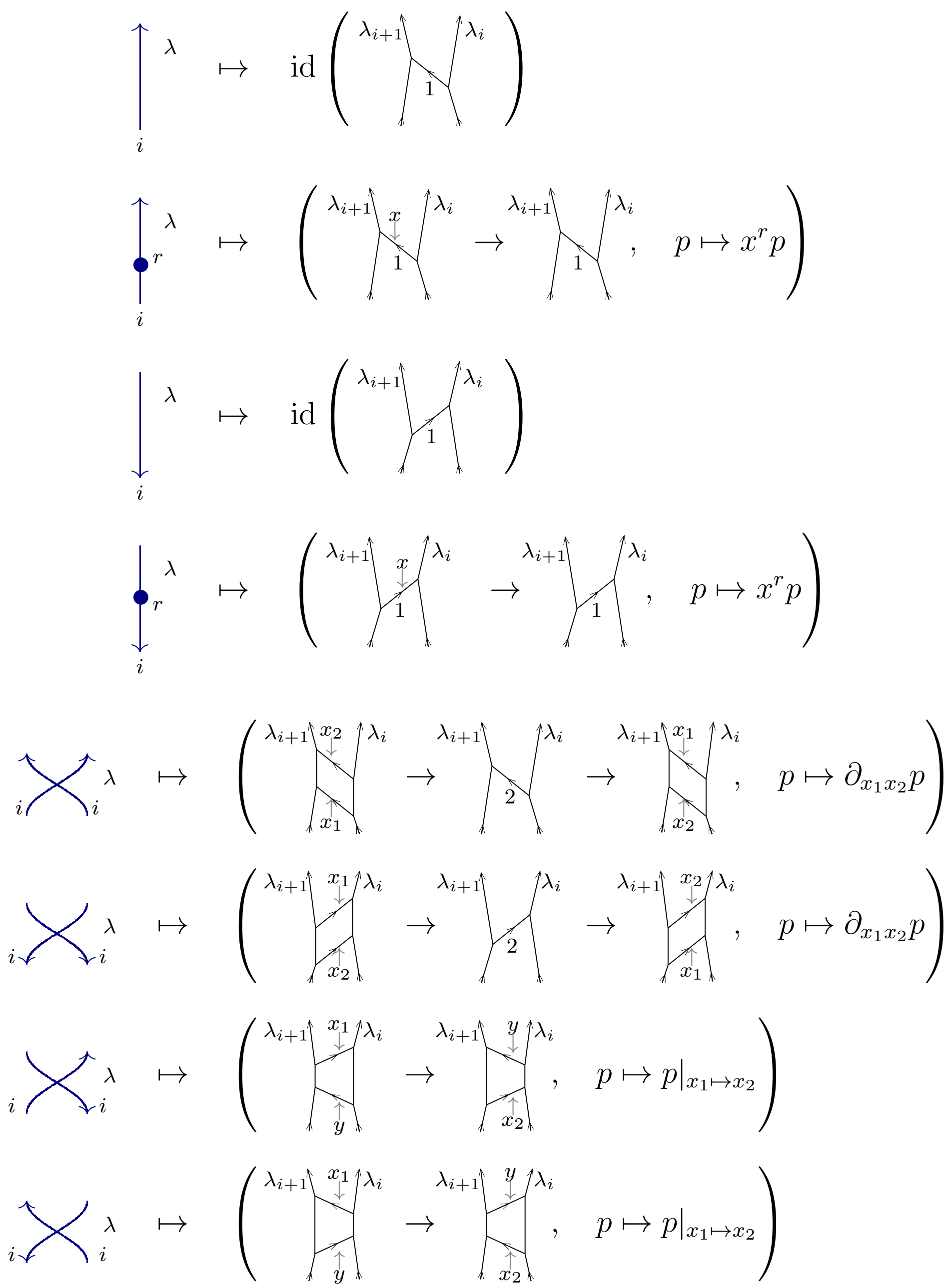
(4.1) $$\partial_{xy} p = \frac{p - p_{|x \leftrightarrow y}}{x - y},$$

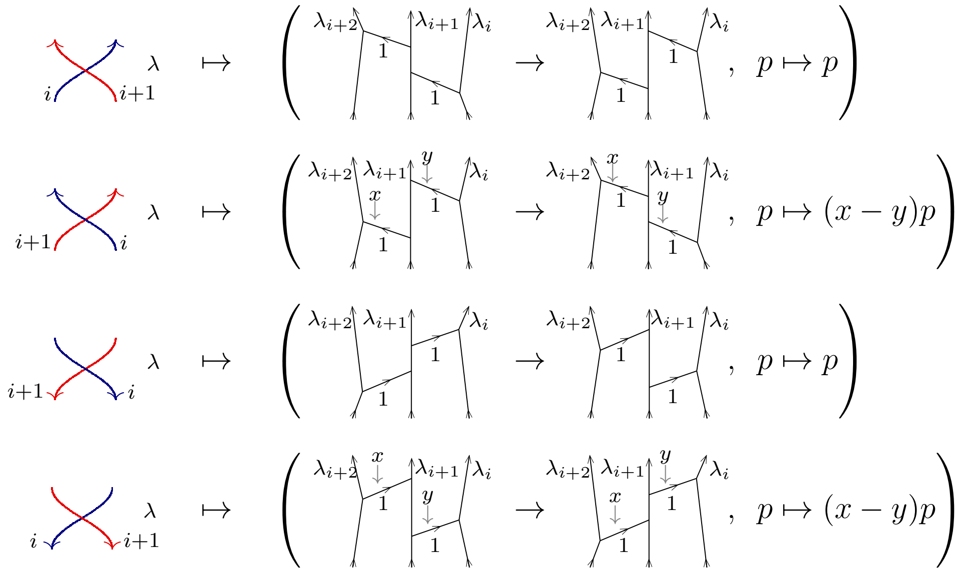
where $p_{|x\leftrightarrow y}$ is the polynomial obtained from p by swapping the variables x and y. Moreover, for $\underline{x} = (x_1, \dots, x_a)$, we use the shorthand notation

(4.2) $$\partial_{\underline{x}y} = \partial_{x_1 y}\partial_{x_2 y} \cdots \partial_{x_a y}$$

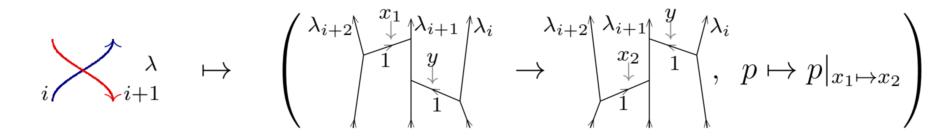
(4.3) $$\partial_{y\underline{x}} = \partial_{y x_1}\partial_{y x_2} \cdots \partial_{y x_a}.$$

Before listing the definition of $\mathcal{F}_{Bim}$, we explain the notation we are using. We denote a bimodule map as a pair, the first term showing the corresponding MOY diagrams (of the source and target 1-morphism), and the second being an explicit formula of the map in terms of the (classes of) polynomials that are the elements of the corresponding rings. In a few cases we have added an intermediate MOY-diagram, in order to clarify the definition. Finally, in order to simplify the pictures, in each formula we only draw the strands that are affected, while on the others we just set the identity. Also in every line we require that the polynomial rings corresponding to the top (respectively bottom) end strands are the same throughout the movie. Furthermore, we only write explicitly the variables of the strands that are relevant in the definition of the corresponding bimodule map.

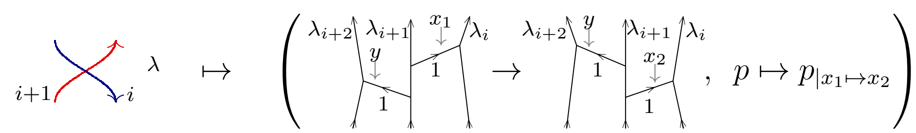


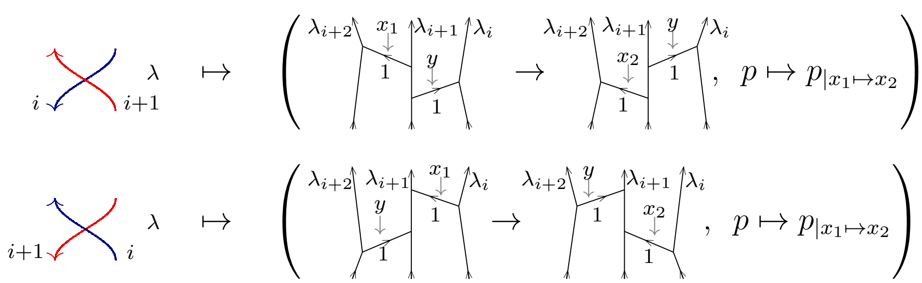


(4.4)

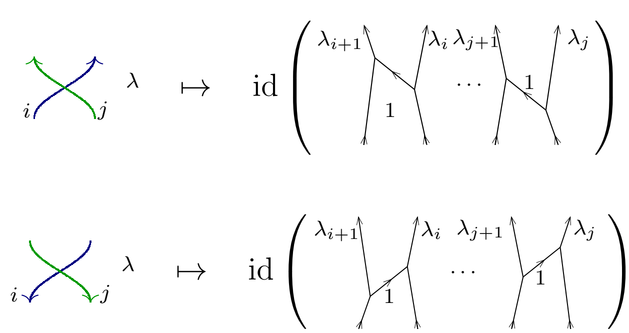


(4.5)

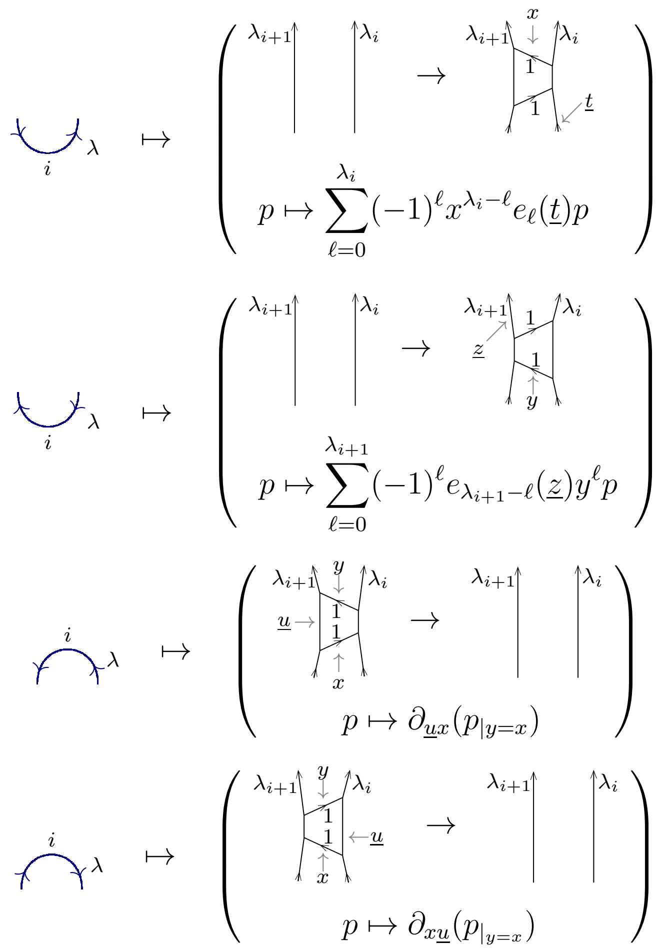




For $|i-j| \geq 2$:



Sideways crossings for $|i-j| \geq 2$ are defined in the same way as in the case of $|i-j| = 1$.



This ends the definition of $\mathcal{F}_{Bim}$. Without giving any details, we remark that the bimodule maps above can be obtained as composites of elementary ones, called *zip, unzip, associativity, digon creation* and *annihilation*, which can be found in [25].

4.3. $\mathcal{F}_{Bim}$ **is a** 2**-functor.** We are now able to explain the relation between our $\mathcal{F}_{Bim}$ and Khovanov and Lauda's (see Subsection 6.3 in [16])

$$\Gamma_d^G\colon \mathcal{U}(\mathfrak{s}l_n)^* \to \mathbf{EqFLAG}_d^* \subset \mathbf{Bim}^*.$$

In the first place, we categorify the homomorphism $\psi_{n,d}$ from Section 2. Note that all the relations in $\mathcal{S}(n,d)$ only depend on $\mathfrak{sl}_n$-weights, except the value of the degree zero bubbles, which truly depend on $\mathfrak{gl}_n$-weights.

Definition 4.1. We define a 2-functor

$$\Psi_{n,d}\colon \mathcal{U}(\mathfrak{s}l_n) \to \mathcal{S}(n,d).$$

On objects and 1-morphisms $\Psi_{n,d}$ is defined just as $\psi_{n,d}\colon \dot{\mathbf{U}}(\mathfrak{sl}_n) \to \dot{\mathbf{S}}(n,d)$ in (2.3). On 2-morphisms we define $\Psi_{n,d}$ as follows. Let D be a string diagram representing a 2-morphism in $\mathcal{U}(\mathfrak{s}l_n)$ (from now on we will simply say that D is a diagram in $\mathcal{U}(\mathfrak{s}l_n)$). Then $\Psi_{n,d}$ maps D to the same diagram, multiplied by a power of -1 depending on the left cups and caps in D according to the rule in (3.1). The labels in $\mathbb{Z}^{n-1}$ of the regions of D are mapped by $\varphi_{n,d}$ to labels in $\mathbb{Z}^n$ of the corresponding regions of $\Psi_{n,d}(D)$, or to $*$. This means that, if D has a region labeled by λ

such that $\varphi_{n,d}(\lambda) \notin \Lambda(n,d)$, then $\Psi_{n,d}(D) = 0$ by definition. Finally, extend this definition to all 2-morphisms by linearity.

It is easy to see that $\Psi_{n,d}$ is well-defined, full and essentially surjective.

In the second place, recall that there is a well-known isomorphism

$$\mathbb{Q}[x_1, \ldots, x_d]^{S_{\lambda_1} \times \cdots \times S_{\lambda_n}} \cong H_{GL(d)}(Fl(\underline{k})),$$

with $\underline{k} = (k_0, k_1, k_2, k_3, \ldots, k_n) = (0, \lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3, \ldots, d)$, for any $\lambda \in \Lambda(n,d)$ (see (6.25) in [16], for example). Using this isomorphism, it is straightforward to check that the following lemma holds by comparing the images of the generators. Recall that Γ_d^G kills all diagrams with labels outside $\Lambda(n,d)$.

Lemma 4.2. The following triangle is commutative

$$\begin{tikzcd} \mathcal{U}(\mathfrak{s}l_n)^* \arrow[rr, "\Gamma_d^G"] \arrow[dr, "\Psi_{n,d}"'] & & \mathbf{Bim}^* \\ & \mathcal{S}(n,d)^* \arrow[ur, "\mathcal{F}_{Bim}"'] & \end{tikzcd}$$

The following result is now an immediate consequence of Khovanov and Lauda's Theorem 6.13.

Proposition 4.3. $\mathcal{F}_{Bim}$ defines a 2-functor from $\mathcal{S}(n,d)^*$ to $\mathbf{Bim}^*$.

One could of course prove Proposition 4.3 by hand. We will just give two sample calculations. The result of the second one, the image of the dotted bubbles, will be needed in a later section.

4.3.1. *Examples of the direct proof of Proposition 4.3.* We shall give the proof for the zig-zag relation of biadjointness and compute the images of the bubbles by $\mathcal{F}_{Bim}$.

Before proceeding, we give some useful relations that are used in the computations. First of all, both the kernel and the image of the divided difference operator ∂_{xy} consist of the polynomials that are symmetric in the variables x and y. If p is symmetric in the variables x and y then

$$\partial_{xy}(p\,q) = p\,\partial_{xy}q$$

for any polynomial q. Also, note that $\partial_{yx} = -\partial_{xy}$.

We shall frequently use the following useful identities (see for example [10] for the proofs). For $\underline{x} = (x_1, \ldots, x_k)$, let $h_j(\underline{x})$ denote the j-th complete symmetric polynomial in the variables $x_1, \ldots, x_k$. Then we have

$$\partial_{y\underline{x}}(y^N) = h_{N-k}(y, \underline{x}), \tag{4.6}$$

and

$$\sum_{j=0}^{k} (-1)^j e_j(\underline{x}) h_{k-j}(\underline{x}) = \delta_{k,0}. \tag{4.7}$$

Moreover, if x, $\underline{u} = (u_1, \ldots, u_a)$ and $\underline{t} = (t_1, \ldots, t_{a+1})$ are variables such that

$$e_l(x, \underline{u}) = e_l(\underline{t}), \quad l = 1, \ldots, a+1,$$

then for every $l = 1, \ldots, a+1$, we have

$$e_l(\underline{u}) = \sum_{j=0}^{l} (-1)^j x^j e_{l-j}(\underline{t}), \tag{4.8}$$

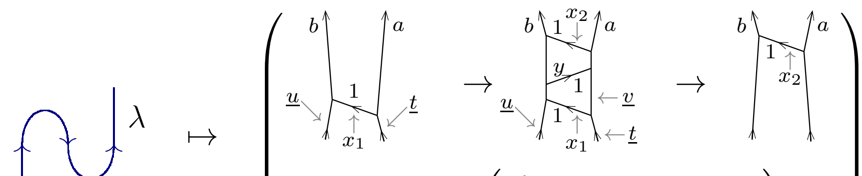
and

$$e_l(\underline{t}) = e_l(\underline{u}) + x e_{l-1}(\underline{u}). \tag{4.9}$$

• The zig-zag relations.

In order to reduce the number of subindices (to keep the notation as concise as possible), we denote $\lambda_i = a$ and $\lambda_{i+1} = b$.

Then the left hand side of the first of the relations (3.2) is mapped by $\mathcal{F}_{Bim}$ as follows:



$$p \mapsto \partial_{x_1\underline{v}}\left(\sum_{\ell=0}^{a}(-1)^\ell x_2^{a-\ell} e_\ell(\underline{v}) p\right)$$

Note that $p = p(x_1, \underline{u}, \underline{t})$ is symmetric in the variables $\underline{u}$ and $\underline{t}$ separately. Also, the lowest trivalent vertex on the right strand in the middle picture of the movie, implies that $e_l(x_1, \underline{v}) = e_l(\underline{t})$, for every $l = 1, \ldots, a+1$. So, x_1^j for $j > a$ is a symmetric polynomial in the variables $\underline{t}$ (e.g. this follows from (4.8) for $l = a+1$). Thus we can write p as:

$$p = \sum_{j=0}^{a} x_1^j q_j(\underline{u}, \underline{t}), \tag{4.10}$$

where $q_j = q_j(\underline{u}, \underline{t})$, $j = 0, \ldots, a$, are polynomials symmetric in $\underline{u}$ and $\underline{t}$ separately.

Then we have:

$$\partial_{x_1\underline{v}}(\sum_{l=0}^{a}(-1)^l x_2^{a-l} e_l(\underline{v}) p) = \sum_{l=0}^{a}(-1)^l x_2^{a-l} \partial_{x_1\underline{v}}(e_l(\underline{v}) \sum_{j=0}^{a} x_1^j q_j) =_{(l \mapsto a-l)}$$

$$= \sum_{l=0}^{a}\sum_{j=0}^{a}(-1)^{a-l} x_2^l q_j \partial_{x_1\underline{v}}(x_1^j e_{a-l}(\underline{v})). \tag{4.11}$$

Since $e_l(x_1, \underline{v}) = e_l(\underline{t})$, for every $l = 1, \ldots, a+1$, by (4.8) we have $e_{a-l}(\underline{v}) = \sum_{k=0}^{a-l}(-1)^k x_1^k e_{a-l-k}(\underline{t})$. After replacing this in (4.11), we get

$$
\begin{aligned}
&= \sum_{l=0}^{a}\sum_{j=0}^{a} x_2^l q_j \sum_{k=0}^{a-l}(-1)^{a-l-k} e_{a-l-k}(\underline{t})\partial_{x_1\underline{v}}(x_1^{j+k}) =_{(4.6)}\\
&= \sum_{l=0}^{a}\sum_{j=0}^{a} x_2^l q_j \sum_{k=0}^{a-l}(-1)^{a-l-k} e_{a-l-k}(\underline{t}) h_{j+k-a}(x_1,\underline{v}) =\\
&= \sum_{l=0}^{a}\sum_{j=0}^{a} x_2^l q_j \sum_{k=0}^{a-l}(-1)^{a-l-k} e_{a-l-k}(\underline{t}) h_{j+k-a}(\underline{t}) =_{(k\mapsto a-l-k)}\\
&= \sum_{l=0}^{a}\sum_{j=0}^{a} x_2^l q_j \sum_{k=0}^{a-l}(-1)^{k} e_{k}(\underline{t}) h_{j-l-k}(\underline{t}).
\end{aligned}
$$

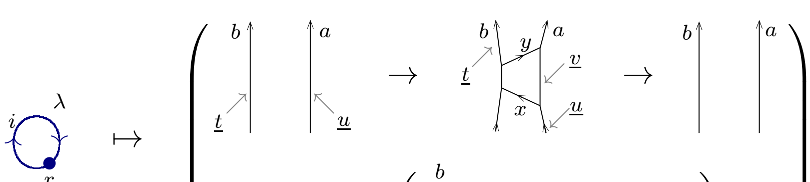
Since $h_p(\underline{t}) = 0$ for $p < 0$, we must have $k \leq j - l (\leq a - l)$ in the innermost summation, and so by (4.7) the last expression above is equal to

$$
\begin{aligned}
&= \sum_{l=0}^{a}\sum_{j=0}^{a} x_2^l q_j \sum_{k=0}^{j-l}(-1)^{k} e_{k}(\underline{t}) h_{j-l-k}(\underline{t}) = \sum_{l=0}^{a}\sum_{j=0}^{a} x_2^l q_j \delta_{j-l,0} =\\
&= \sum_{j=0}^{a} x_2^j q_j = p_{|x_1 \mapsto x_2},
\end{aligned}
$$

which is just the identity map, as wanted.

- Images of bubbles by $\mathcal{F}_{Bim}$.

Again we denote $\lambda_i = a$ and $\lambda_{i+1} = b$.
The clockwise oriented bubble with $r \geq 0$ dots on it is mapped by $\mathcal{F}_{Bim}$ as follows



$$p \mapsto \partial_{x\underline{v}}\left(\sum_{\ell=0}^{b}(-1)^{\ell} e_{b-\ell}(\underline{t}) x^{\ell+r} p\right)$$

The polynomial $p = p(\underline{t}, \underline{u})$ is symmetric in the variables $\underline{t}$ and $\underline{u}$ separately. In particular, we have $\partial_{x\underline{v}}(p\,q) = p\,\partial_{x\underline{v}}(q)$, for any polynomial q. We have:

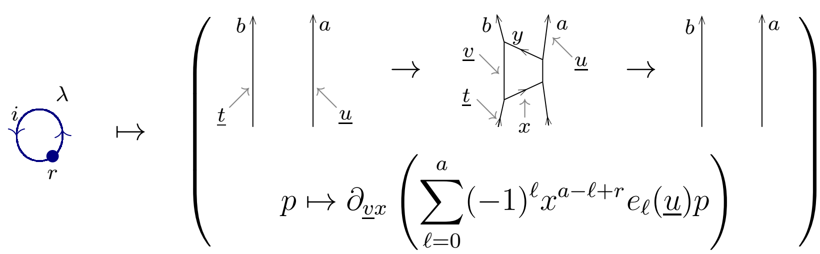
$$\begin{aligned}\partial_{x\underline{v}}\Big(\sum_{l=0}^{b}(-1)^l e_{b-l}(\underline{t})x^{l+r}p\Big) &= \sum_{l=0}^{b}(-1)^l e_{b-l}(\underline{t})p\partial_{x\underline{v}}(x^{l+r}) =_{(4.6)}\\ &= p\sum_{l=0}^{b}(-1)^l e_{b-l}(\underline{t})h_{l+r-a+1}(\underline{u}) =_{(l\mapsto b-l)}\\ &= p(-1)^b\sum_{l=0}^{b}(-1)^l e_l(\underline{t})h_{b-a+r+1-l}(\underline{u}).\end{aligned}$$

Since $e_l(\underline{t}) = 0$ for $l > b$, and $h_{b-a+r+1-l}(\underline{u}) = 0$, for $l > b-a+r+1$, we have that the clockwise oriented bubble is mapped by $\mathcal{F}_{Bim}$ to the following bimodule map:

$$p \mapsto p(-1)^b \sum_{l=0}^{b-a+r+1}(-1)^l e_l(\underline{t})h_{b-a+r+1-l}(\underline{u}). \tag{4.12}$$

In particular, if $b-a+r+1 < 0$, i.e. if $r < a-b-1$, the bubble is mapped to zero, and if $r = a-b-1$, the bubble is mapped to $(-1)^b$ times the identity (note that $a-b = \lambda_i - \lambda_{i+1}$ is $\mathfrak{sl}_n$ weight). Also, r can be naturally extended to $r \geq a-b-1$ (in (4.12)), i.e. to include fake bubbles in the case $a \leq b$.

The counter-clockwise oriented bubble with $r \geq 0$ dots on it is mapped by $\mathcal{F}_{Bim}$ as follows



$$p \mapsto \partial_{\underline{v}x}\left(\sum_{\ell=0}^{a}(-1)^\ell x^{a-\ell+r}e_\ell(\underline{u})p\right)$$

Completely analogously as above, we have that the counter-clockwise oriented bubble is mapped by $\mathcal{F}_{Bim}$ to the following bimodule map:

$$p \mapsto p(-1)^{b+1}\sum_{l=0}^{a-b+r+1}(-1)^l e_l(\underline{u})h_{a-b+r+1-l}(\underline{t}). \tag{4.13}$$

Again, from $r < b-a-1$, the bubble is mapped to zero, and if $r = b-a-1$, it is mapped to $(-1)^{b+1}$ times the identity. Moreover, r can be naturally extended to $r \geq b-a-1$, i.e. to include fake bubbles in the case $b \leq a$.

Remark 4.4. Our reason for changing the signs from [16], was to make the signs in the image of the degree zero bubbles, i.e. $(-1)^b$ for the clockwise bubble and $(-1)^{b+1}$ for the counter-clockwise bubble, coincide with those of (3.9).

Finally, by the Giambelli and the dual Giambelli formulas (see e.g. [10]), from (4.12) and (4.13) the infinite Grassmannian relation follows directly.

5. Comparisons with $\mathcal{U}(\mathfrak{sl}_n)$

In this section we show the analogues for $\mathcal{S}(n,d)$ of some of Khovanov and Lauda's results on the structure of $\mathcal{U}(\mathfrak{s}l_n)$. Our results are far from complete. More work will need to be done to understand the structure of $\mathcal{S}(n,d)$ better.

To simplify terminology, by a 2-functor we will always mean an additive $\mathbb{Q}$-linear degree preserving 2-functor.

5.1. **Categorical inclusions and projections.** In the first place, we categorify the homomorphisms $\pi_{d',d}$ from Section 2.

Definition 5.1. Let $d' = d + kn$, with $k \in \mathbb{N}$. We define a 2-functor

$$\Pi_{d',d} \colon \mathcal{S}(n,d') \to \mathcal{S}(n,d).$$

On objects and 1-morphisms $\Pi_{d',d}$ is defined as $\pi_{d',d}$. On 2-morphisms $\Pi_{d',d}$ is defined as follows. For any diagram D in $\mathcal{S}(n,d')$ with regions labeled $\lambda \in \Lambda(n,d')$ such that $\lambda - (k^n) \in \Lambda(n,d)$, let $\Pi_{d',d}(D)$ be given by the same diagram with labels of the form $\lambda - (k^n)$, multiplied by $(-1)^k$ for every left cap and left cup in D. For any other diagram D, let $\Pi_{d',d}(D) = 0$. Extend this definition to all 2-morphisms by linearity.

Note that $\Pi_{d',d}$ is well-defined, because $\overline{\lambda} = \overline{\lambda - (k^n)}$. The extra $(-1)^k$ for left cups and caps is necessary to match our normalization of the degree zero bubbles. It also ensures that we have

$$\Pi_{d',d}\Psi_{n,d'} = \Psi_{n,d},$$

where

$$\Psi_{n,d} \colon \mathcal{U}(\mathfrak{s}l_n) \to \mathcal{S}(n,d)$$

is the 2-functor defined in Definition 4.1.

Note also that the $\Pi_{d',d}$ form something like an inverse system of 2-functors between 2-categories, because

$$\Pi_{d',d}\Pi_{d'',d'} = \Pi_{d'',d}$$

(compare to (2.4)). We say "something like" an inverse system, because we have not been able to find a precise definition of such a structure in the literature on n-categories. Also one would have to think carefully if the "inverse limit" of the $\mathcal{S}(n,d)$ would still be Krull-Schmidt. Finally, there appears to be no general theorem that says that the Grothendieck group of an inverse limit is the inverse limit of the Grothendieck groups (even for algebras there is no such theorem). So we cannot (yet) reasonably conjecture the categorification of the embedding (2.5). All we can say at the moment is the following:

Corollary 5.2. We have:

(1) Let $f1_\alpha$ be a 2-morphism in $\mathcal{U}(\mathfrak{s}l_n)$. Let $d_0 > 0$ be the minimum value such that $\alpha = \overline{\beta}$ with $\beta \in \Lambda(n,d_0)$. Then $f = 0$ if and only if $\Psi_{n,d_0+nk}(f) = 0$ for any $k \geq 0$.

(2) Let $\{f_i 1_\alpha\}_{i=1}^s$ be a finite set of 2-morphisms in $\mathrm{Hom}_{\mathcal{U}(\mathfrak{sl}_n)}(x,y)$. Then the $f_i 1_\alpha$ are linearly independent if and only if there exists a $d > 0$ such that the $\Psi_{n,d}(f_i 1_\lambda)$ are linearly independent in $\mathrm{Hom}_{\mathcal{S}(n,d)}(x,y)$.

The proof of Corollary 5.2 follows from Khovanov and Lauda's Lemma 6.16 in [16], which implies Theorem 1.3 in [16], our Lemma 4.2 and the remarks above Corollary 5.2.

The main reason for trying to categorify (2.5) is the following: if the inverse limit of the $\mathcal{S}(n,d)$ turns out to exist, perhaps it contains a sub-2-category which categorifies $\mathbf{U}_q(\mathfrak{sl}_n)$.

5.2. **The structure of the 2HOM-spaces.** We now turn our attention to the structure of the 2HOM-spaces in $\mathcal{S}(n,d)$. The reader should compare our results to Khovanov and Lauda's in [16]. We first show the analogue of Lemma 6.15.

5.2.1. *Bubbles for* $n = 2$. For starters suppose that $n = 2$. Let $\lambda = (a,b) \in \Lambda(2,d)$. Recall that a partially symmetric polynomial $p(\underline{x},\underline{y}) = p(\underline{x},\underline{y}) \in \mathbb{Q}[x_1,\ldots,x_a,y_1,\ldots,y_b]^{S_a\times S_b}$ is called *supersymmetric* if the substitution $x_1 = t = y_1$ gives a polynomial independent of t (see [11] and [23] for example). We let $R^{ss}_{a,b}$ denote the ring of supersymmetric polynomials. The *elementary* supersymmetric polynomials are

$$e_j(\underline{x},\underline{y}) = \sum_{s=0}^{j}(-1)^s h_{j-s}(\underline{x})\varepsilon_s(\underline{y}),$$

where $h_{j-s}(\underline{x})$ is the $j-s$th complete symmetric polynomial in a variables and $\varepsilon_s(\underline{y})$ the sth elementary symmetric polynomial in b variables, which we put equal to zero if $s > b$ by convention. It is easy to see that $e_j(\underline{x},\underline{y})$ is supersymmetric, because we have

$$\prod_{r=1}^{a}\prod_{s=1}^{b}\frac{1-y_rZ}{1-x_sZ} = \sum_j e_j(\underline{x},\underline{y})Z^j.$$

Using the supersymmetric analogue of the Giambelli formula we can define the *supersymmetric Schur polynomials*

$$\pi_\alpha(\underline{x},\underline{y}) = \det(e_{\alpha_i+j-i}(\underline{x},\underline{y}))$$

for $1 \leq i,j \leq m$ and α a partition of length m. In the following lemma we give the basic facts about supersymmetric Schur polynomials, which are of interest to us in this paper. For the proofs see [11, 23] and the references therein. Let $\Gamma(a,b)$ be the set of partitions α such that $\alpha_j \leq b$ for all $j > a$.

Lemma 5.3. We have

(1) If $\alpha \notin \Gamma(a,b)$, then $\pi_\alpha(\underline{x},\underline{y}) = 0$.
(2) The set $\{\pi_\alpha(\underline{x},\underline{y}) \mid \alpha \in \Gamma(a,b)\}$ is a linear basis of $R^{ss}_{a,b}$.
(3) We have

$$\pi_\alpha(\underline{x},\underline{y})\pi_\beta(\underline{x},\underline{y}) = \sum_\gamma C^\gamma_{\alpha\beta}\pi_\gamma(\underline{x},\underline{y}),$$

where $C^\gamma_{\alpha\beta}$ are the Littlewood-Richardson coefficients.

(4) We have

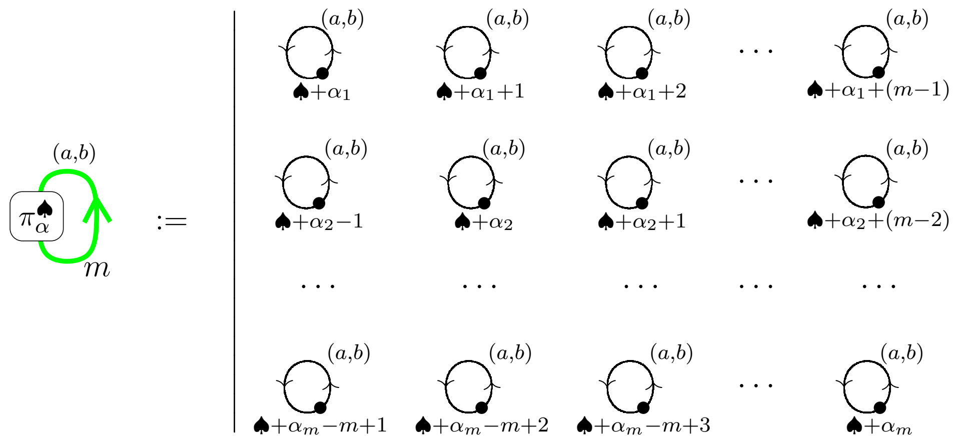
$$\pi_\alpha(\underline{x},\underline{y}) = (-1)^{|\alpha|}\pi_{\alpha'}(\underline{y},\underline{x}),$$

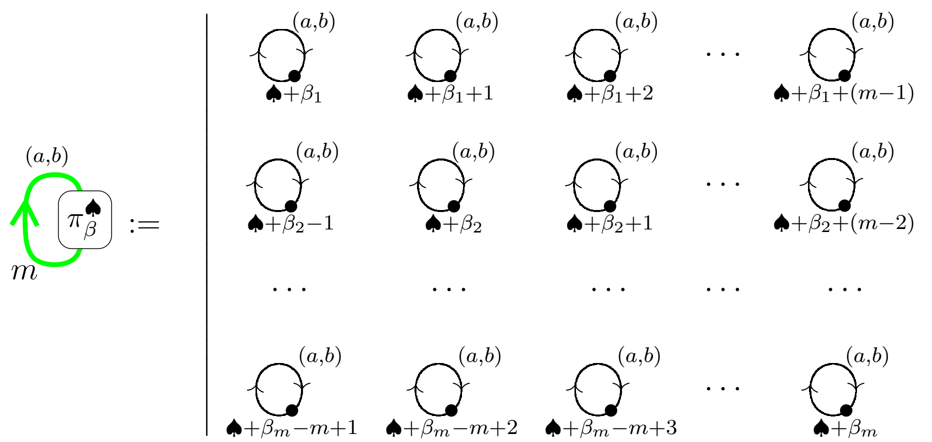
where $|\alpha| = \sum_i \alpha_i$ and α' is the conjugate partition.

(5) We also get the ordinary Schur polynomials as special cases

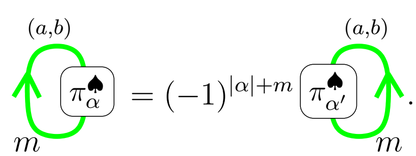
$$\pi_\alpha(\underline{x},0) = \pi_\alpha(\underline{x})$$
$$\pi_\beta(0,\underline{y}) = (-1)^{|\beta|}\pi_{\beta'}(\underline{y}).$$

In [18] the *extended calculus* in $\mathcal{U}(\mathfrak{sl}_2)$ was developed. Here we only use a little part of it. Below, for partitions α, β with length m, we write $\alpha^{\spadesuit} = \alpha - (a-b) - m$ for counter-clockwise oriented bubbles of thickness m in a region labeled (a,b), and $\beta^{\spadesuit} = \beta + (a-b) - m$ for clockwise oriented bubbles of thickness m. Recall that thick bubbles labeled by a spaded Schur polynomial can be written as Giambelli type determinants (see Equations (3.33) and (3.34) in [18], but bear our sign conventions in mind):

(5.1) 

(5.2) 

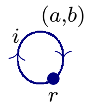
The reader unfamiliar with [18] can interpret the above simply as definitions. In Proposition 4.10 in [18] it is proved that the clockwise thick bubbles form a linear basis of $\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(1_{a-b})$ and that they obey the Littlewood-Richardson rule under multiplication. Of course the counter-clockwise thick bubbles form another basis and also obey the L-R rule. Proposition 4.10 in [18] also shows the relation between the two bases (recall that we have slightly different sign conventions in this paper and that α' is the partition conjugate to α):

(5.3) 

Therefore, in our case the non-zero clockwise thick bubbles also form a nice basis of $\mathrm{END}_{\mathcal{S}(n,d)}(1_{(a,b)})$.

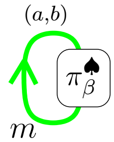
Lemma 5.4. $\mathcal{F}_{Bim}\colon \mathrm{END}_{\mathcal{S}(n,d)}(1_{(a,b)}) \to R^{ss}_{a,b}$ is a ring isomorphism, mapping the clockwise thick bubbles to the corresponding supersymmetric Schur polynomials.

Proof. It is clear that the thick bubbles generate $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{(a,b)})$, because they are the image of the thick bubbles in $\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_\mathrm{n})}(1_{a-b})$, which form a linear basis. Since $\Psi_{n,d}$ is a 2-functor, we see that the multiplication of bubbles in $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{(a,b)})$ satisfies the Littlewood-Richardson rule. In Section 4 we showed that using $\mathcal{F}_{Bim}$ we get



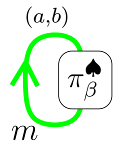
$$\mapsto \quad (-1)^b e_{-(a-b)+1+r}(\underline{x},\underline{y}).$$

This implies that



$$\mapsto \quad (-1)^{mb}\pi_\beta(\underline{x},\underline{y}).$$

Therefore, by Lemma 5.3, all we have to show is that



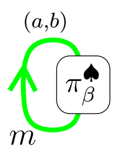
$$= \quad 0$$

if $\beta \notin \Gamma(a,b)$. We proceed by induction on m. Note that if $m < a+1$, then $\beta \in \Gamma(a,b)$, so the induction starts at $m = a+1$. If $m = a+1$, then $\beta_{a+1} = \beta_m > b$ implies that $\beta_i > b$ holds for all $i = 1,\ldots,m$, because β is a partition. Therefore, for any $i = 1,\ldots,m$, we have

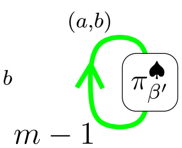
$$\beta_i + a - b - m = \beta_i + a - b - (a+1) = \beta_i - b - 1 \geq 0.$$

Thus the bubble is real and equals zero because its inner region is labeled $(-1, a+b+1) \notin \Lambda(2,d)$.

Suppose that $m > a+1$ and that the result has been proved for bubbles of thickness $< m$. Using induction, we will prove that it holds for bubbles of thickness m. The trouble is that in this case the bubble can be fake, so we cannot repeat the argument above. Instead we use a second induction, this time on β_m. Write $\beta' = (\beta_1,\ldots,\beta_{m-1})$. First suppose $\beta_m = 0$. Then

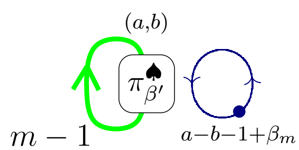


$$= \quad (-1)^b$$



$$= \quad 0$$

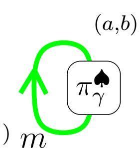
by induction on m. Now suppose $\beta_m > 0$. Then we have



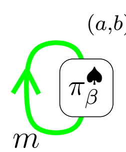
$$= \quad 0$$

by induction on m. By Pieri's rule, the left-hand side equals

$$\sum_{\beta<\gamma\leq\beta+(\beta_m)}$$

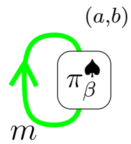


$$+$$



where $\beta + (\beta_m) = (\beta_1 + \beta_m, \beta_2, \ldots, \beta_{m-1}, 0)$. Note that for any $\beta < \gamma \leq \beta + (\beta_m)$, we have $\gamma \notin \Gamma(a,b)$ and $\gamma_m < \beta_m$. Thus, by induction on β_m, all the thick bubbles labeled with $\pi^{\spadesuit}_{\gamma}$ are

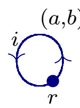
zero. This implies that



$$= \quad 0.$$

$\square$

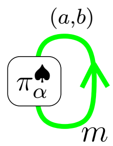
Note that for bubbles with the opposite orientation we have



$$\mapsto \quad (-1)^{b+1} e_{(a-b)+1+r}(y,x).$$

This implies that

(5.4)



$$\mapsto \quad (-1)^{m(b+1)} \pi_{\alpha}(y,x).$$

This way we get another isomorphism between $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{(a,b)})$ and $R^{ss}_{a,b}$.

5.2.2. *Bubbles for* $n > 2$. For $n > 2$, we get polynomials in thick bubbles of $n-1$ colors. Unfortunately we have not been able to find anything in the literature on a generalization of supersymmetric polynomials to more than two alphabets. Nor has the extended calculus for $\mathcal{U}(\mathfrak{s}l_n)$ been worked out and written up for $n > 2$ so far. Therefore all we can say is the following. Let

$$S\Pi_{\lambda} = \bigotimes_{i=1}^{n-1} R^{ss}_{\lambda_i,\lambda_{i+1}}.$$

There is a surjective homomorphism

$$S\Pi_{\lambda} \to \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\lambda})$$

sending supersymmetric polynomials to the corresponding clockwise oriented thick bubbles. Note that $\Psi_{n,d}\colon \Pi_{\overline{\lambda}} \cong \mathrm{END}_{\mathcal{U}(\mathfrak{sl}_\mathrm{n})}(1_{\overline{\lambda}}) \to \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\lambda})$ factors through $S\Pi_{\lambda}$. Recall that

$$\Pi_{\overline{\lambda}} \cong \bigotimes_{i=1}^{n-1} \Lambda(\underline{x}),$$

where $\Lambda(\underline{x})$ is the ring of symmetric functions in infinitely many variables $\underline{x} = (x_1, x_2, \ldots)$ (see (3.24) and Lemma 6.15 in [16]). The map $\Pi_{\overline{\lambda}} \to S\Pi_{\lambda}$ referred to above is defined by

$$\pi^i_{\alpha}(\underline{x}) \mapsto \pi_{\alpha}(\underline{x}, \underline{y}),$$

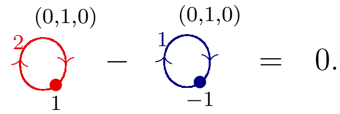
where $(\underline{x}, \underline{y}) = (x_1, \ldots, x_{\lambda_i}, y_1, \ldots, y_{\lambda_{i+1}})$ and $\pi^i_{\alpha}(\underline{x}) = 1 \otimes \cdots \otimes \pi_{\alpha}(\underline{x}) \otimes \cdots \otimes 1$ belongs to the i-th tensor factor and .

Note also that the projection

$$S\Pi_{\lambda} \to \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\lambda})$$

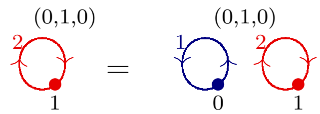
is not an isomorphism in general. For example, with blue bubbles colored 1 and red bubbles colored 2, we have

(5.5)

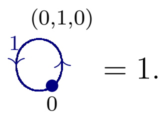


To see why this holds, first use

(5.6)

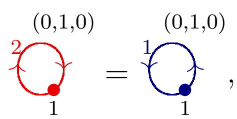


This equation holds because



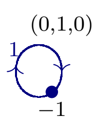
Then slide the red bubble inside the blue one on the r.h.s. of (5.6) with bubble-slide (3.25). Note that we have to switch the colors i and $i+1$ in (3.25), but that only changes the sign on the r.h.s. of that bubble-slide, as remarked below the list of bubble-slides. After doing that, only one blue bubble with one dot survives, because in the interior of that bubble, which is labeled $(1,0,0)$, only a degree zero red bubble is non-zero. This holds because the red bubbles of positive degree are real bubbles and their interior is labeled $(1,-1,1) \notin \Lambda(3,1)$. The degree zero red bubble is equal to 1, by (3.9). Thus we have obtained

(5.7)



which is equal to

(5.8)



by the infinite Grassmannian relation (3.13) and relation (3.9).

The relation above between bubbles of different colors generalizes. Using the extended calculus for $\mathcal{S}(n,d)$ [18], we can see that whenever λ is of the form $(\ldots,0,\lambda_i,0,\ldots)$, bubbles of the same degree of colors $i-1$ and i are equal up to a sign. This also has to do with the fact that compositions of d of the form $(\ldots,a,0,\ldots)$ and $(\ldots,0,a,\ldots)$ are equivalent as objects in the Karoubi envelope $\dot{\mathcal{S}}(n,d)$. We will explain this in Remark 7.1. Here we just leave a conjecture about $\mathrm{End}_{\mathcal{S}(\mathrm{n,d})}(1_\lambda)$.

Conjecture 5.5. Let $\lambda \in \Lambda(n,d)$ be arbitrary and let $\mu \in \Lambda(n,d)$ be obtained from λ by placing all zero entries of λ at the end, but without changing the relative order of the non-zero entries, e.g. for $\lambda = (2,0,1)$ we get $\mu = (2,1,0)$. Then we conjecture that

$$\mathrm{End}_{\mathcal{S}(\mathrm{n,d})}(1_\lambda) \cong S\Pi_\mu.$$

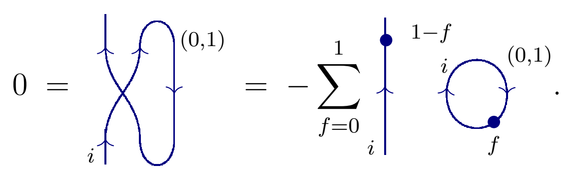
Note that if $\mu_k \neq 0$ and $\mu_{k+1} = 0$ for a certain $1 \leq k \leq n-1$ in Conjecture 5.5, then $S\Pi_\mu$ is isomorphic to the algebra of all partially symmetric polynomials $\mathbb{Q}[x_1,\ldots,x_d]^{S_{\mu_1}\times\cdots\times S_{\mu_k}}$. This follows from the fact that $R^{ss}_{\mu_k,0}$ is the algebra of symmetric polynomials in μ_k variables. For example, suppose $\mu = (1,1,0)$. Then $R^{ss}_{1,1} \cong \mathbb{Q}[x-y]$ and $R^{ss}_{1,0} \cong \mathbb{Q}[y]$, so $S\Pi_{(1,1,0)} \cong \mathbb{Q}[x-y]\otimes\mathbb{Q}[y]$. The latter algebra is isomorphic to $\mathbb{Q}[x,y]$ by

$$(x-y)\otimes 1 + 1\otimes y \leftrightarrow x, \qquad 1\otimes y \leftrightarrow y.$$

5.2.3. *More general 2-morphisms.* There is not all that much that we know about more general 2-hom spaces in $\mathcal{S}(n,d)$. Let us give a conjecture about an "analogue" of Lemma 3.9 from [16] for $\mathcal{S}(n,d)$. Let $\nu \in \mathbb{N}[I]$ and $\mathbf{i},\mathbf{j} \in \nu$. Recall (see Section 2 in [14] and Subsection 3.2.2 in [16]) that ${}_{\mathbf{i}}R(\nu)_{\mathbf{j}}$ is the vector space of upwards oriented braid-like diagrams as in $\mathcal{U}(\mathfrak{s}l_n)$ whose lower boundary is labeled by $\mathbf{i}$ and upper boundary by $\mathbf{j}$, modulo the braid-like relations in $\mathcal{U}(\mathfrak{s}l_n)$. Note that all strands of such a diagram have labels uniquely determined by $\mathbf{i}$ and $\mathbf{j}$. Note also that the braid-like relations in $\mathcal{U}(\mathfrak{s}l_n)$ are independent of the weights, so the definition of ${}_{\mathbf{i}}R(\nu)_{\mathbf{j}}$ does not involve weights. Unfortunately, we cannot define the analogue of ${}_{\mathbf{i}}R(\nu)_{\mathbf{j}}$ for $\mathcal{S}(n,d)$, because there the braid-like diagrams with a region labeled by a weight outside $\Lambda(n,d)$ are equal to zero, creating a weight dependence. However, we will be able to use ${}_{\mathbf{i}}R(\nu)_{\mathbf{j}}$ and the fact that $\Psi_{n,d}$ is full. Khovanov and Lauda (Lemma 3.9, Definition 3.15 and the remarks thereafter, and Theorem 1.3 in [16]) showed that the obvious map

$$\Psi_{\mathbf{i},\mathbf{j},\overline{\lambda}}\colon {}_{\mathbf{i}}R(\nu)_{\mathbf{j}} \otimes \Pi_{\overline{\lambda}} \to \mathrm{HOM}_{\mathcal{U}(\mathfrak{sl}_n)}(\mathcal{E}_{\mathbf{i}}1_{\overline{\lambda}}, \mathcal{E}_{\mathbf{j}}1_{\overline{\lambda}})$$

is an isomorphism. Unfortunately it is also impossible to factor $\mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\mathbf{i}}1_{\lambda}, \mathcal{E}_{\mathbf{j}}1_{\lambda})$ so nicely into braid-like diagrams and bubbles. For example, let $\lambda = (0,1)$ and look at the following reduction to bubble relation



This result generalizes to any λ, using the extended calculus in [18]. Thus, given any λ, there is an upper bound t_r for the number of dots on the arcs of the r-strands. Any braid-like diagram in $\mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\mathbf{i}}1_{\lambda}, \mathcal{E}_{\mathbf{j}}1_{\lambda})$ with more than t_r dots on an r-colored strand can be written as a linear combination of braid-like diagrams whose r-strands have $\leq t_r$ dots with coefficients in $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\lambda})$. By the fullness of $\Psi_{n,d}$ and the fact that ${}_{\mathbf{i}}R(\nu)_{\mathbf{j}}$ has a basis ${}_{\mathbf{i}}B_{\mathbf{j}}$ which only contains a finite number of braid-like diagrams if one forgets the dots (see Theorem 2.5 in [14]), it follows that $\mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\mathbf{i}}1_{\lambda}, \mathcal{E}_{\mathbf{j}}1_{\lambda})$ is finitely generated over $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\lambda})$. In Section 6 we will say a little more about the image of

$$B_{\mathbf{i},\mathbf{j},\overline{\lambda}} = \Psi_{\mathbf{i},\mathbf{j},\overline{\lambda}}({}_{\mathbf{i}}B_{\mathbf{j}}) \subseteq \mathrm{HOM}_{\mathcal{U}(\mathfrak{sl}_n)}(\mathcal{E}_{\mathbf{i}}1_{\overline{\lambda}}, \mathcal{E}_{\mathbf{j}}1_{\overline{\lambda}})$$

in $\mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\mathbf{i}}1_{\lambda}, \mathcal{E}_{\mathbf{j}}1_{\lambda})$ under $\Psi_{n,d}$. Recall again that ${}_{\mathbf{i}}B_{\mathbf{j}}$ is Khovanov and Lauda's basis of ${}_{\mathbf{i}}R(\nu)_{\mathbf{j}}$ in Theorem 2.5 in [14]. Unfortunately, all we can give for now is a conjecture.

Conjecture 5.6. We conjecture that $\mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\mathbf{i}}1_{\lambda}, \mathcal{E}_{\mathbf{j}}1_{\lambda})$ is a free right module of finite rank over $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\lambda})$.

Note that if $\mathcal{E}_{\mathbf{i}}1_{\lambda} = 1_{\mu}\mathcal{E}_{\mathbf{i}}$ and $\mathcal{E}_{\mathbf{j}}1_{\lambda} = 1_{\mu}\mathcal{E}_{\mathbf{j}}$, then we also conjecture that $\mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(1_{\mu}\mathcal{E}_{\mathbf{i}}, 1_{\mu}\mathcal{E}_{\mathbf{j}})$ is a free left module of finite rank over $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\mu})$. However, it is not hard to give examples which show that, if the conjectures are true at all, the ranks of $\mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(1_{\mu}\mathcal{E}_{\mathbf{i}}1_{\lambda}, 1_{\mu}\mathcal{E}_{\mathbf{j}}1_{\lambda})$ as a right $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\lambda})$-module and as a left $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\mu})$-module are not equal in general. This is not surprising, because the graded dimensions of $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\lambda})$ and $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_{\mu})$ are not equal in general either.

5.3. **The categorical anti-involution.** The last part of this section is dedicated to the categorification of the anti-involution $\tau\colon \dot{\mathrm{S}}(n,d) \to \dot{\mathrm{S}}(n,d)^{\mathrm{op}}$ in Section 2. We simply follow Khovanov and Lauda's Subsection 3.3.2. Let $\mathcal{S}(n,d)^{\mathrm{coop}}$ denote the 2-category which the same objects as $\mathcal{S}(n,d)$, but with the directions of the 1- and 2-morphisms reversed. We define a strict degree preserving 2-functor $\widetilde{\tau}\colon \mathcal{S}(n,d) \to \mathcal{S}(n,d)^{\mathrm{coop}}$ by

$$\begin{aligned} \lambda &\mapsto \lambda \\ 1_\mu \mathcal{E}_{s_1}\mathcal{E}_{s_2}\cdots\mathcal{E}_{s_{m-1}}\mathcal{E}_{s_m}1_\lambda\{t\} &\mapsto 1_\lambda \mathcal{E}_{-s_m}\mathcal{E}_{-s_{m-1}}\cdots\mathcal{E}_{-s_2}\mathcal{E}_{-s_1}1_\mu\{-t+t'\} \\ \zeta &\mapsto \zeta^*. \end{aligned}$$

Let D be a diagram, then D^* is obtained from D by rotating the latter $180°$. Since $\mathcal{S}(n,d)$ is cyclic, it does not matter in which way you rotate. By linear extension this defines ζ^* for any 2-morphism. The shift t' is defined by requiring that $\widetilde{\tau}$ be degree preserving. One can easily check that $\widetilde{\tau}$ is well-defined. For more details on the analogous $\widetilde{\tau}$ defined on $\mathcal{U}(\mathfrak{sl}_n)$ see Subsection 3.3.2 in [16]. As a matter of fact $\widetilde{\tau}$ is a functorial anti-involution. The most important result about $\widetilde{\tau}$ is the analogue of Remark 3.20 in [16].

Lemma 5.7. There are degree zero isomorphisms of graded $\mathbb{Q}$-vector spaces

$$\begin{aligned} \mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(fx,y) &\cong \mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(x,\widetilde{\tau}(f)y) \\ \mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(xg,y) &\cong \mathrm{HOM}_{\mathcal{S}(\mathrm{n,d})}(x,y\widetilde{\tau}(g)), \end{aligned}$$

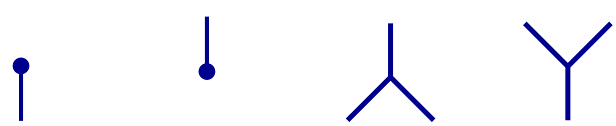
for any 1-morphisms x, y, f, g.

6. The diagrammatic Soergel categories and $\mathcal{S}(n,d)$

6.1. **The diagrammatic Soergel category revisited.** In this subsection we recall the diagrammatics for Soergel categories introduced by Elias and Khovanov in [8]. Actually we first recall the version sketched by Elias and Khovanov in Section 4.5 and used by Elias and Krasner in [9]. After that we will comment on how to alter it in order to get the original version by Elias and Khovanov. Note that both versions categorify the Hecke algebra, although they are not equivalent as categories. In this paper we will need both versions.

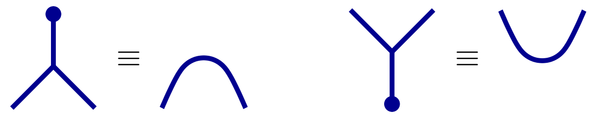
Fix a positive integer n. The category $\mathcal{SC}_1(n)$ is the category whose objects are finite length sequences of points on the real line, where each point is colored by an integer between 1 and $n-1$. We read sequences of points from left to right. Two colors i and j are called *adjacent* if $|i-j|=1$ and *distant* if $|i-j|>1$. The morphisms of $\mathcal{SC}_1(n)$ are given by generators modulo relations. A morphism of $\mathcal{SC}_1(n)$ is a $\mathbb{Q}$-linear combination of planar diagrams constructed by horizontal and vertical gluings of the following generators (by convention no label means a generic color j):

- Generators involving only one color:

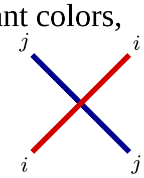


EndDot StartDot Merge Split

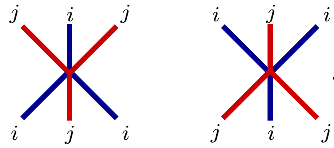
It is useful to define the cap and cup as



- Generators involving two colors:
  - The 4-valent vertex, with distant colors,



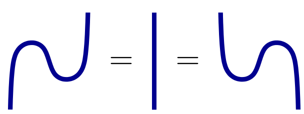
  - and the 6-valent vertex, with adjacent colors i and j

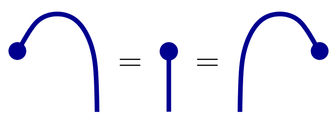


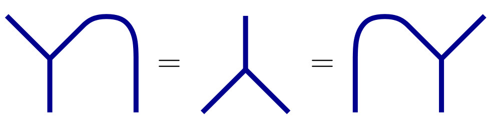
In this setting a diagram represents a morphism from the bottom boundary to the top. We can add a new colored point to a sequence and this endows $\mathcal{SC}_1(n)$ with a monoidal structure on objects, which is extended to morphisms in the obvious way. Composition of morphisms consists of stacking one diagram on top of the other.

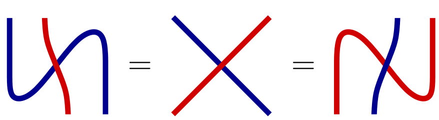
We consider our diagrams modulo the following relations.

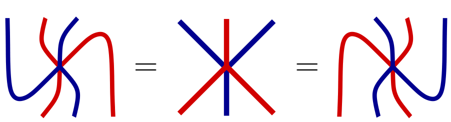
"Isotopy" relations:

(6.1) 

(6.2) 

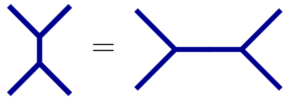
(6.3) 

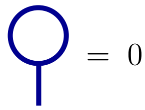
(6.4) 

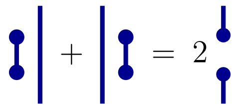
(6.5) 

The relations are presented in terms of diagrams with generic colorings. Because of isotopy invariance, one may draw a diagram with a boundary on the side, and view it as a morphism in $\mathcal{SC}_1(n)$ by either bending strands up or down. By the same reasoning, a horizontal line corresponds to a sequence of cups and caps.

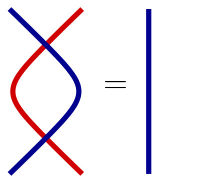
One color relations:

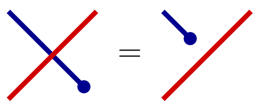
(6.6) 

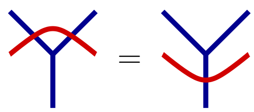
(6.7) 

(6.8) 

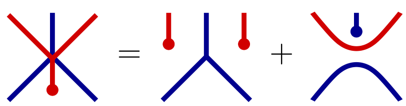
Two distant colors:

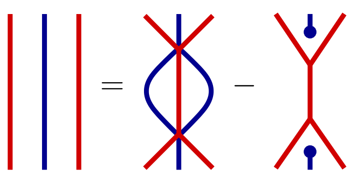
(6.9) 

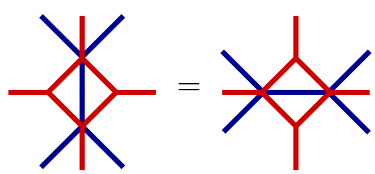
(6.10) 

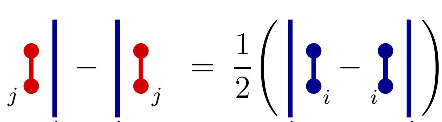
(6.11) 

Two adjacent colors:

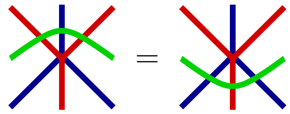
(6.12) 

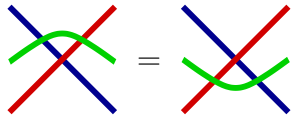
(6.13) 

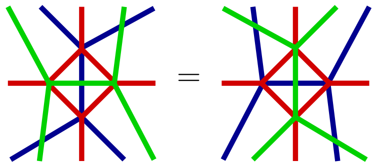
(6.14) 

(6.15) 

Relations involving three colors: (adjacency is determined by the vertices which appear)

(6.16) 

(6.17) 

(6.18) 

Introduce a grading on $\mathcal{SC}_1(n)$ by declaring dots to have degree 1, trivalent vertices degree -1 and 4- and 6-valent vertices degree 0.

Definition 6.1. The category $\mathcal{SC}_2(n)$ is the category containing all direct sums and grading shifts of objects in $\mathcal{SC}_1(n)$ and whose morphisms are the grading preserving morphisms from $\mathcal{SC}_1(n)$.

Definition 6.2. The category $\mathcal{SC}(n)$ is the Karoubi envelope of the category $\mathcal{SC}_2(n)$.

6.2. **The extension $\mathcal{SC}'(n)$ of $\mathcal{SC}(n)$.** In [8] Elias and Khovanov give a slightly different diagrammatic Soergel category, denoted $\mathcal{SC}'(n)$, which is a faithful extension of $\mathcal{SC}(n)$. The objects of $\mathcal{SC}'_1(n)$ are the same as those of $\mathcal{SC}_1(n)$. The vector spaces of morphisms are an extension of the ones of $\mathcal{SC}_1(n)$ in the following sense. Regions can be decorated with boxes colored by i for $1 \leq i \leq n$, which we depict as

$$\boxed{i}$$

For f a polynomial in the set of boxes colored from 1 to n we use the shorthand notation

$$\boxed{f}$$

The set of boxes is therefore in bijection with the polynomial ring in n variables. Let s_i be the transposition that switches i and $i+1$. Define the formal symbol

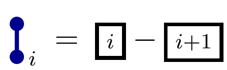
$$\boxed{\partial_i f} = \boxed{\partial_{x_i x_{i+1}} f}$$

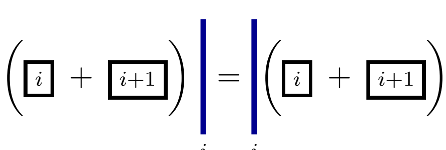
where $\partial_{x_i x_{i+1}}$ was defined in Equation (4.1). This way any box $\boxed{f}$ can be written as

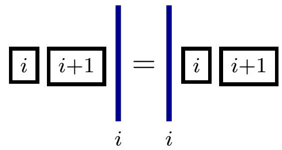
$$\boxed{f} = \boxed{P_i(f)} + \boxed{i}\,\boxed{\partial_i f}$$

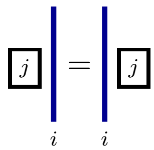
where $P_i(f)$ is a polynomial which is symmetric in $\boxed{i}$ and $\boxed{i{+}1}$ (we will take this formula as a definition of $P_i(f)$).

The boxes are related to the previous calculus by the *box relations*

(6.19) 

(6.20) 

(6.21) 

(6.22)  for $j \neq i, i+1$.

It is clear that $\mathcal{SC}(n)$ is a faithful monoidal subcategory of $\mathcal{SC}'(n)$. As explained in Section 4.5 of [8], the category $\mathcal{SC}(n)$ is also isomorphic to the quotient of $\mathcal{SC}'(n)$ by the central morphism

$$\boxed{e_1} \overset{def}{=} \sum_{i=1}^{n} \boxed{i}.$$

This result depends subtly on the base field, which in our case is $\mathbb{Q}$.

The category $\mathcal{SC}'_1(n)$ has a grading induced by the one of $\mathcal{SC}_1(n)$, if we declare that a box colored i has degree 2 for all $1 \geq i \geq n$.

Definition 6.3. The category $\mathcal{SC}'_2(n)$ is the category containing all direct sums and grading shifts of objects in $\mathcal{SC}'_1(n)$ and whose morphisms are the grading preserving morphisms from $\mathcal{SC}'_1(n)$. The category $\mathcal{SC}'(n)$ is the Karoubi envelope of the category $\mathcal{SC}'_2(n)$.

Elias and Khovanov's main result in [8] is that $\mathcal{SC}(n)$ and $\mathcal{SC}'(n)$ are equivalent to the corresponding Soergel categories. A corollary to that is the following theorem, where K_0 is the split Grothendieck group and $K_0^{\mathbb{Q}(q)}(-) = K_0(-) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$.

Theorem 6.4 (Elias-Khovanov, Soergel)**.** We have

$$K_0^{\mathbb{Q}(q)}(\mathcal{SC}(n)) \cong K_0^{\mathbb{Q}(q)}(\mathcal{SC}'(n)) \cong H_q(n).$$

As explained in [8], this result also depends on the fact that we are working over $\mathbb{Q}$. Recall that $\mathcal{SC}(n)$ and $\mathcal{SC}'(n)$ are monoidal categories, with the monoidal structure defined by concatenation. Therefore their Grothendieck groups are algebras indeed.

Let $\mathbf{Bim}(n)^* = \mathrm{End}_{\mathbf{Bim}^*}(\mathbb{Q}[x_1, \ldots, x_n])$. Elias and Khovanov defined functors from $\mathcal{SC}(n)$ and $\mathcal{SC}'(n)$ to $\mathbf{Bim}(n)^*$ (see [8, 9]) which we denote by $\mathcal{F}_{EK}$ and $\mathcal{F}'_{EK}$ respectively.

6.3. **A functor from $\mathcal{SC}(n)$ to $\mathcal{S}(n,n)^*((1^n),(1^n))$.** Let $n \geq 1$ be arbitrary but fixed. In this subsection we define an additive $\mathbb{Q}$-linear monoidal functor

$$\Sigma_{n,n} \colon \mathcal{SC}_1(n) \to \mathcal{S}(n,n)^*((1^n),(1^n)),$$

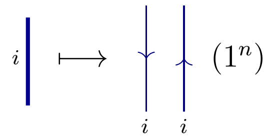
where the target is the monoidal category whose objects are the 1-endomorphisms of (1^n) in $\mathcal{S}(n,n)^*$ and whose morphisms are the 2-morphisms between such 1-morphisms in $\mathcal{S}(n,n)^*$. This monoidal functor categorifies the homomorphism $\sigma_{n,n}$ from Section 2.

On objects: $\Sigma_{n,n}$ sends the empty sequence in $\mathcal{SC}_1(n)$ to $1_n = 1_{(1^n)}$ in $\mathcal{S}(n,n)^*$ and the one-term sequence (i) to $\mathcal{E}_{-i}\mathcal{E}_{+i}1_n$, with $\Sigma_{n,n}(jk)$ given by the horizontal composite $\mathcal{E}_{-j}\mathcal{E}_{+j}\mathcal{E}_{-k}\mathcal{E}_{+k}1_n$.

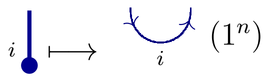
On morphisms:

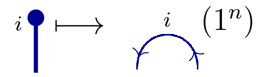
- The empty diagram is sent to the empty diagram in the region labeled (1^n).

- The vertical line coloured i is sent to the identity 2-morphism on $\mathcal{E}_{-i}\mathcal{E}_{+i}1_n$.

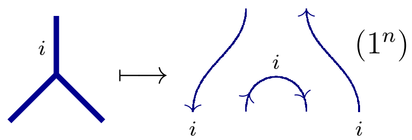


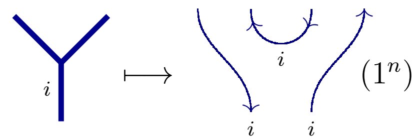
- The *StartDot* and *EndDot* morphisms are sent to the cup and the cap respectively:



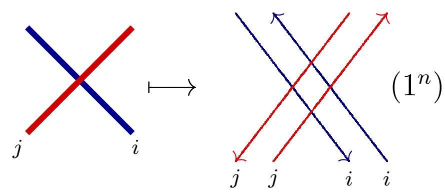


- *Merge* and *Split* are sent to diagrams involving cups and caps:



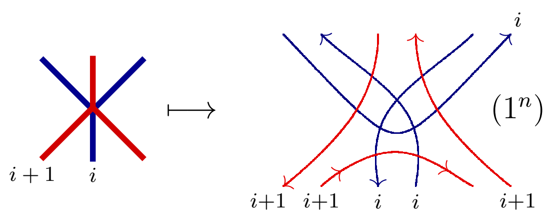


- The *4-valent vertex* with distant colors. For i and j distant we have:

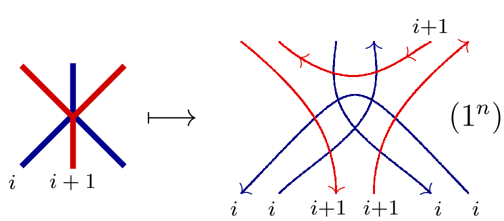


- For the *6-valent vertices* we have:

(6.23)



and



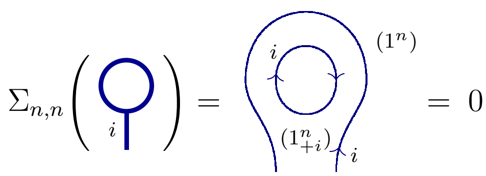
It is clear that $\Sigma_{n,n}$ respects the gradings of the morphisms. Moreover, let us remark that, in the decategorified picture, the image of $H_q(n)$ lies in the projection of the zero weight space of $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ onto $\dot{\mathbf{S}}(n,n)$, so we have $E_iE_{-i} = E_{-i}E_i$. Using the 2-isomorphism $\mathcal{E}_i\mathcal{E}_{-i} \cong \mathcal{E}_{-i}\mathcal{E}_i$ given by the crossing, we obtain a 2-functor naturally isomorphic to $\Sigma_{n,n}$. However, this 2-functor cannot be obtained by simply inverting the orientation of the diagrams defining $\Sigma_{n,n}$, as can be easily checked. As a matter of fact, inverting the orientations does not even give a 2-functor, e.g. relation (6.12) is not preserved.

Lemma 6.5. $\Sigma_{n,n}$ is a monoidal functor.

Proof. The assignment given by $\Sigma_{n,n}$ clearly respects the monoidal structures of the categories $\mathcal{SC}_1(n)$ and $\mathrm{End}_{\mathcal{S}(n,n)^*}(1^n)$. So we only need to show that $\Sigma_{n,n}$ is a functor, i.e. it respects the relations (6.1) to (6.18).

"Isotopy relations": Relations (6.1) to (6.5) are straightforward to check and correspond to isotopies of their images under $\Sigma_{n,n}$.

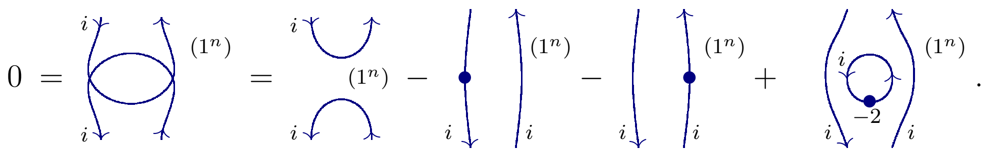
One color relations: To check the one color relations we only need to use the $\mathfrak{sl}_2$ relations. Relation (6.6) corresponds to an easy isotopy of diagrams in $\mathcal{S}(n,n)$. For relation (6.7) we have



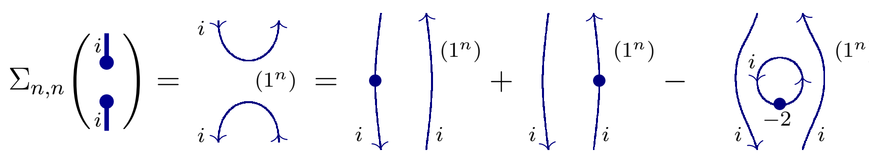
because the bubble in the diagram on the r.h.s. has negative degree. We have used the notation $1^n_{+i} = (1,\ldots,2,0,1,\ldots,1)$, with the 2 on the ith coordinate.

Relation (6.8) requires some more work. First notice that from relations (3.9) and (3.11) it follows that

(6.24)

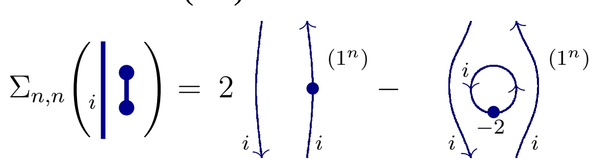


The first diagram is zero, because the middle region has label $(1,\ldots,3,-1,\ldots,1) \notin \Lambda(n,n)$, with 3 on the ith coordinate. Therefore



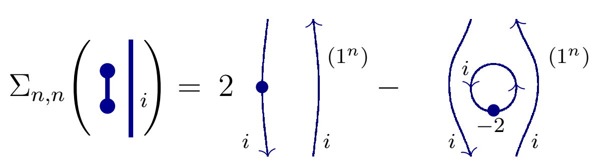
Using (3.22) and the bubble evaluation (3.9) we obtain

(6.25)

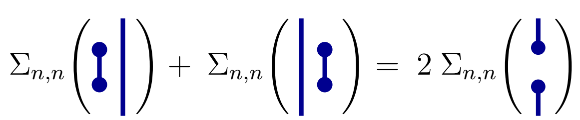


and

(6.26)

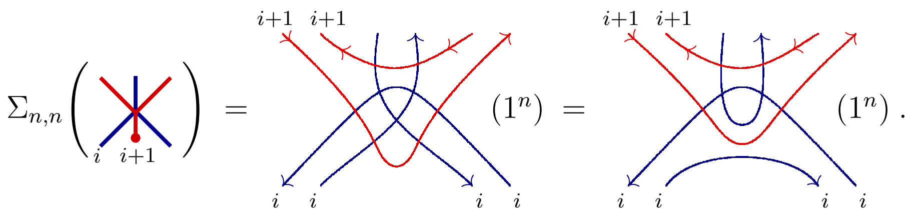


This establishes that



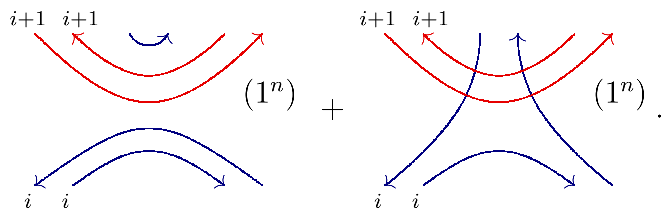
Two distant colors: Checking relations (6.9) to (6.11) is straightforward and only uses relations (3.16) and (3.17) with distant colors i and j.

Adjacent colors: To prove relation (6.12) we first notice that using (3.20) we get

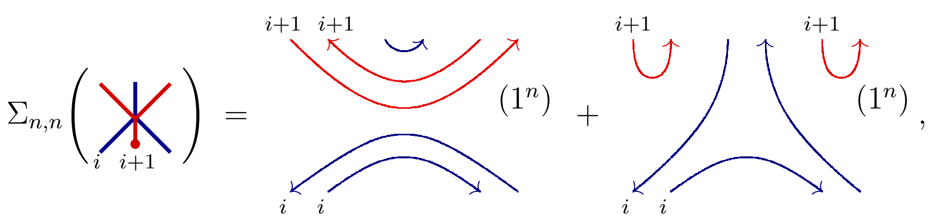


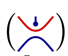
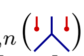
Note that the other term on the r.h.s. of (3.20) is equal to zero, because it contains a region whose label has a negative entry, i.e. does not belong to $\Lambda(n,n)$.

Using (3.11) followed by (3.16) and (3.9) gives

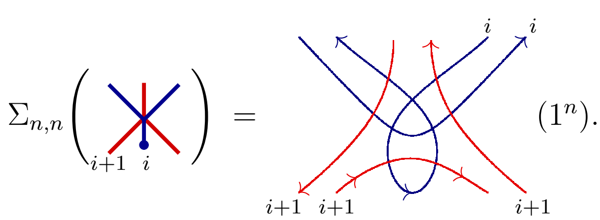


Applying (3.12) to the two red strands in the middle region of the second term (only one term survives) followed by (3.16) and (3.9) gives

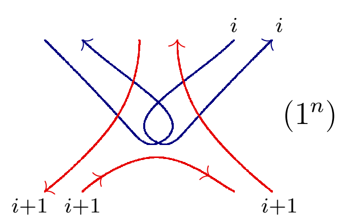


which is equal to $\Sigma_{n,n}$  $+ \Sigma_{n,n}$  .

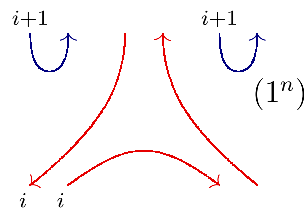
The corresponding relation with colors switched is not difficult to prove. We have



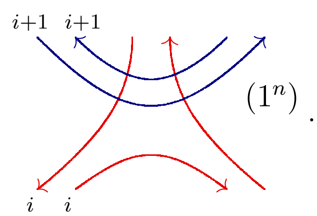
Use (3.17) on the bottom part of the diagram. Only one of the resulting terms survives (use the first relation in (3.14)), which in turn equals



(use the first relation in (3.14) combined with (3.15)). Applying (3.11) we get two terms, one of which is

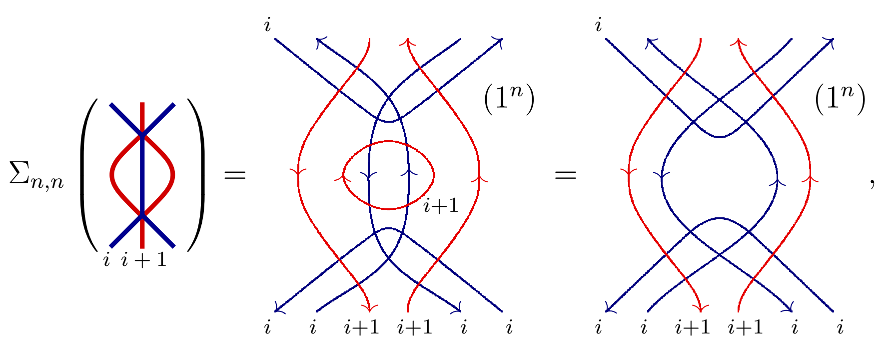


(this follows easily from (3.16)) and the other equals



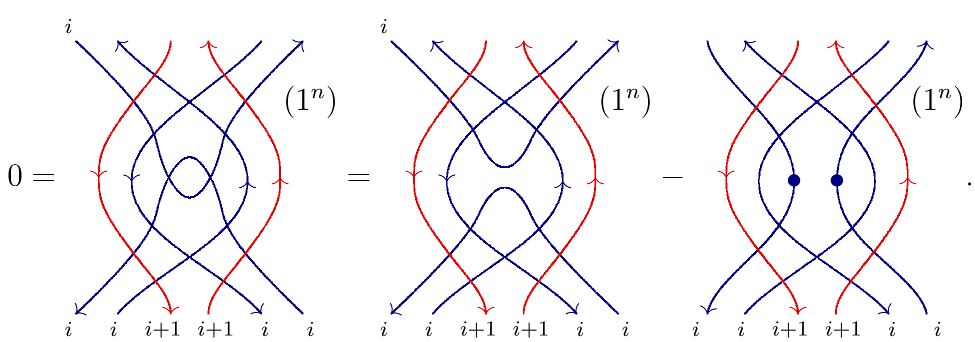
Here we used (3.9). The rest of the computation is the same as in the previous case.

We now prove relation (6.13). We only prove the case where ”blue“ corresponds to i and ”red“ corresponds to $i+1$. The relation with the colors reversed is proved in the same way. Start with



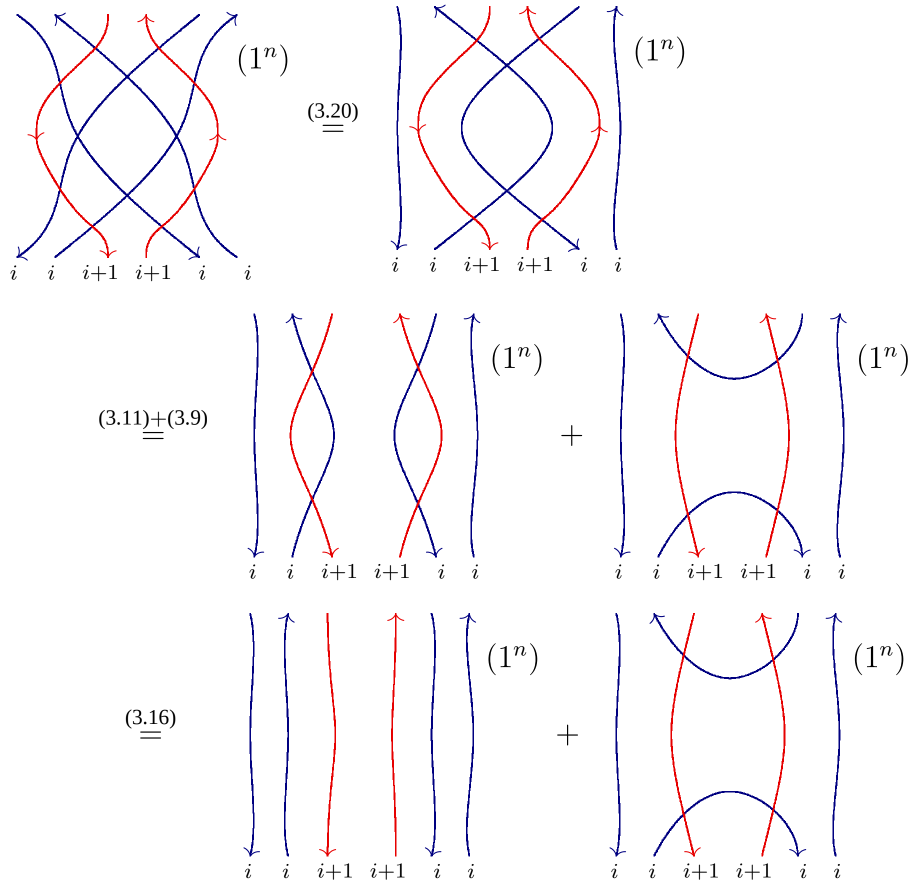
where the second equality follows from (3.16) and (3.9). Now notice that

(6.27)

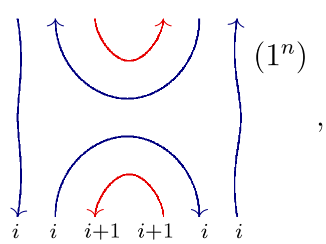


The first equality in (6.27) comes from the fact that the inner most region of the diagram has a label outside $\Lambda(n,n)$. The second equality follows from (3.11). The last term is the only non-zero term coming from the sum in (3.11) (this is a consequence of (3.14)).

Applying (3.14) and (3.15) to the last term, we obtain a diagram that can be simplified further by successive application of (3.20), (3.11), (3.16) and again (3.9).

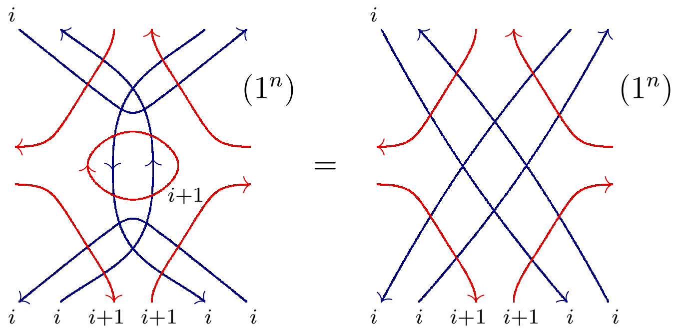


Applying (3.12) to the vertical red strands in the second term, followed by (3.16) and (3.9), we get that it is equal to



which equals $\Sigma_{n,n}\big(\big)$. Therefore $\Sigma_{n,n}\big(\big) = \Sigma_{n,n}\big(\big) + \Sigma_{n,n}\big(\big)$.

We now prove relation (6.14). We denote the left and right hand sides of (6.14) L and R, respectively. We have

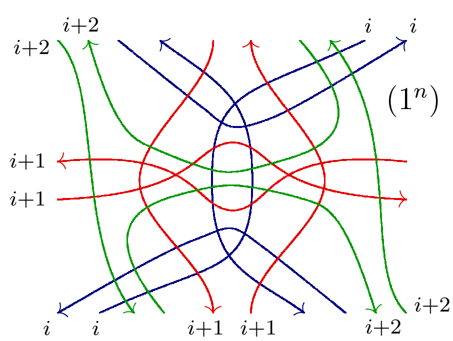
$\Sigma_{n,n}(L) =$ 

The second equality is obtained as in Equation (6.27). The same argument shows that this equals $\Sigma_{n,n}(R)$.

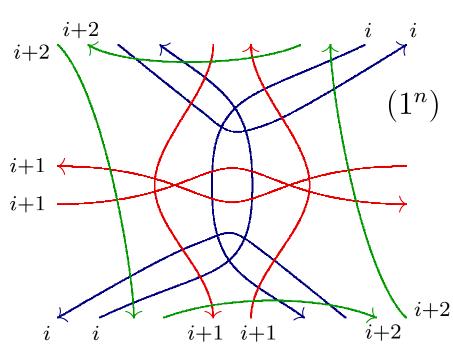
Relation (6.15) is straightforward to check (it only uses bubble slides).

Relations involving three colors: Relations (6.16) and (6.17) are easy because the green strands have to be distant from red and blue and so we have all Reidemeister 2 and Reidemeister 3 like moves between green and one of the other colors.

It remains to prove that $\Sigma_{n,n}$ respects relation (6.18). First notice that the diagrams on the left- and right-hand side of (6.18) are invariant under $180°$ rotations and that they can be obtained from one another using a $90°$ rotation. Therefore it suffices to show that the image of one of them is invariant under $90°$ rotations. Denote by L the diagram on the left-hand-side of (6.18). Then

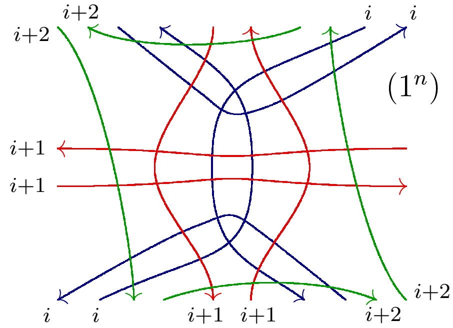
$\Sigma_{n,n}(L) =$  .

Taking into account that the green strands are distant from the blue ones, we apply (3.16) and a sequence of Reidemeister 3 like moves to obtain

$\Sigma_{n,n}(L) =$  .

Using (3.27) twice between the two horizontal red lines and a vertical blue line, followed by (3.12) gives

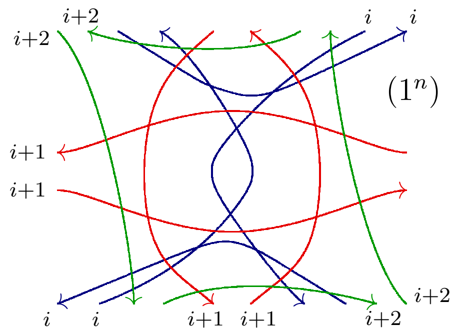
$$\Sigma_{n,n}(L) \;=$$



.

Notice that the sums in (3.12) are not increasing and therefore there are no terms with dots here. Applying (3.27) and (3.28) to the top and bottom we can pass the top and bottom (i, i) crossings to the middle of the diagram (the terms coming from the sums in (3.28) are zero). We get

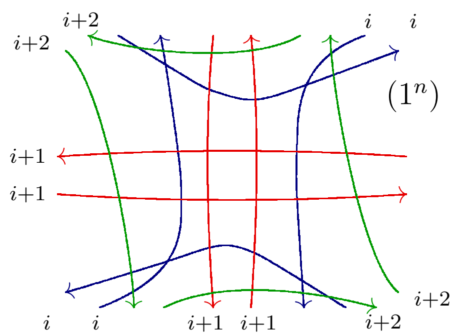
$$\Sigma_{n,n}(L) \;=$$



.

Using (3.11) in the middle of the diagram followed by (3.16) and a sequence of Reidemeister 3 like moves to pass the vertical red strands to the middle gives

$$\Sigma_{n,n}(L) \;=$$



,

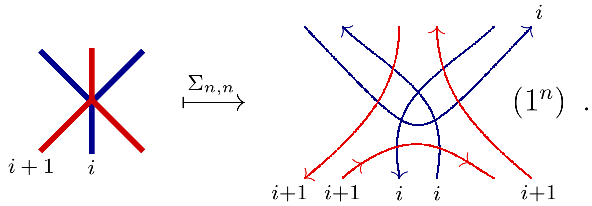
which is symmetric under $90°$ rotations. $\square$

6.4. $\mathcal{SC}_1(n)$ is a full sub-2-category of $\mathcal{S}(n,n)$.

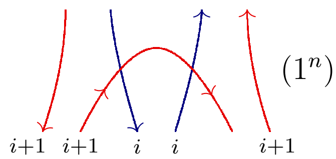
Lemma 6.6. The following diagram commutes

$$\begin{tikzcd}
\mathcal{SC}_1(n) \arrow[rr, "\mathcal{F}_{EK}"] \arrow[dr, "\Sigma_{n,n}"'] & & \mathbf{Bim}(n)^* \\
& \mathcal{S}(n,n)^*((1^n),(1^n)) \arrow[ur, "\mathcal{F}_{Bim}"'] &
\end{tikzcd}$$

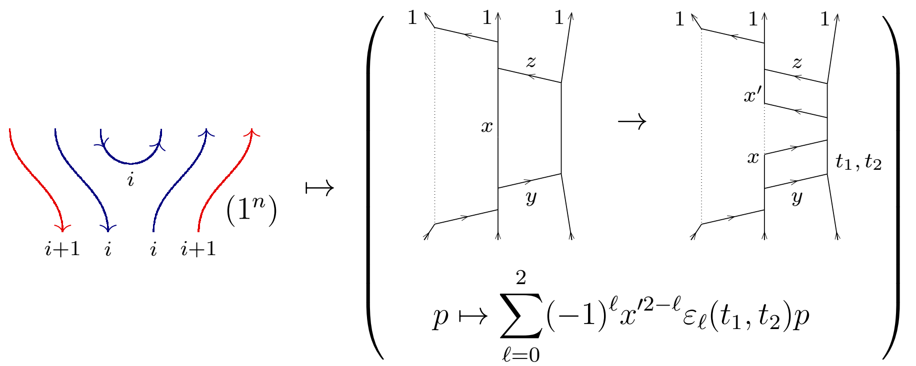
Proof. The commutativity of the diagram can be checked by direct computation. Most of the computation is straightforward except for the 6-valent vertex. To compute its image under $\mathcal{F}_{Bim}\Sigma_{n,n}$ we divide it in layers and compute the bimodule maps for each layer. We do the case with the colors as in Equation 6.23, the other case being similar. Remember that



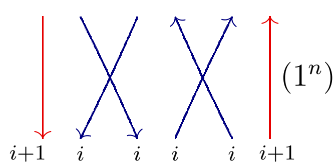
It is easy to see that the map corresponding to the layer



consists only of a relabeling of variables. The next one is



The next step consists of the two crossings between strands labeled i,



corresponding to the map $p \mapsto \partial_{zx'}\partial_{xy}p$. The left pointing (i,i)-crossing and the remaining $(i, i+1)$ crossings consist only of relabeling of variables and shifts. Putting everything together, the reader can check that this map coincides with the one obtained from $\mathcal{F}_{EK}$ by a straightforward computation. □

We now get to the main result of this subsection.

Proposition 6.7. The functor $\Sigma_{n,n}$ is an equivalence of categories.

Proof. We have to show that $\Sigma_{n,n}$ is essentially surjective and fully faithful. By the commutation 2-isomorphisms, i.e. the relations involving Reidemeister II and III type moves between diagrams in $\mathcal{S}(n,n)$, we can commute the factors of any object x in $\mathcal{S}(n,n)^*((1^n),(1^n))$ so that it becomes a direct sum of objects whose factors are all of the form $\mathcal{E}_{-i}\mathcal{E}_{+i}1_n$. This is always possible because

x has to have as many factors $\mathcal{E}_{-j}$ as $\mathcal{E}_{+j}$, for any $j = 1, \ldots, n-1$, or else x contains a factor 1_λ with $\lambda \notin \Lambda(n,n)$ and is therefore equal to zero. This shows that $\Sigma_{n,n}$ is essentially surjective.

Since the functor $\mathcal{F}_{EK}$ is faithful [8], it follows from Lemma 6.6 that $\Sigma_{n,n}$ is faithful too. Therefore it only remains to show that $\Sigma_{n,n}$ is full. To this end we first note that

$$\widetilde{\tau}(\mathcal{E}_{-i}\mathcal{E}_{+i}1_n) = \mathcal{E}_{-i}\mathcal{E}_{+i}1_n.$$

By simply checking the definitions one sees that the natural isomorphisms in Corollary 4.12 in [8] and the ones in Lemma 5.7 in this paper intertwine $\Sigma_{n,n}$. For example, we have a commutative square

$$\begin{tikzcd}
\mathrm{HOM}_{\mathcal{SC}_1(n)}(i\underline{\boldsymbol{k}}, \underline{\boldsymbol{j}}) \arrow[r, "\cong"] \arrow[d, "\Sigma_{n,n}"'] & \mathrm{HOM}_{\mathcal{SC}_1(n)}(\underline{\boldsymbol{k}}, i\underline{\boldsymbol{j}}) \arrow[d, "\Sigma_{n,n}"'] \\
\mathrm{HOM}_{\mathcal{S}(n,n)}(\mathcal{E}_{-i}\mathcal{E}_{+i}1_n\Sigma_{n,n}(\underline{\boldsymbol{k}}), \Sigma_{n,n}(\underline{\boldsymbol{j}})) \arrow[r, "\cong"] & \mathrm{HOM}_{\mathcal{S}(n,n)}(\Sigma_{n,n}(\underline{\boldsymbol{k}}), \mathcal{E}_{-i}\mathcal{E}_{+i}1_n\Sigma_{n,n}(\underline{\boldsymbol{j}})).
\end{tikzcd}$$

This observation together with the results after Corollary 4.12 in Section 4.3 in [8] and the fact that $\Sigma_{n,n}$ is additive and $\mathbb{Q}$-linear implies that it is enough to prove that

$$\Sigma_{n,n}\colon \mathrm{HOM}_{\mathcal{SC}_1(n)}(\emptyset, \underline{\boldsymbol{i}}) \to \mathrm{HOM}_{\mathcal{S}(n,n)}(1_n, \mathcal{E}_{-i_1}\mathcal{E}_{+i_1}\cdots\mathcal{E}_{-i_t}\mathcal{E}_{+i_t}1_n)$$

is surjective, where $\underline{\boldsymbol{i}} = (i_1, \ldots, i_t)$ is a sequence of t points of strictly increasing color $1 \le i_1 < i_2 < \cdots < i_t \le n-1$. If $t = 0$, then this is true, because $\mathrm{HOM}_{\mathcal{SC}_1(n)}(\emptyset, \emptyset) \cong \mathbb{Q}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]$ by Elias and Khovanov's Theorem 1. Note that

$$\mathcal{S}(n,d)^*((1^n),(1^n)) \cong \mathbb{Q}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]$$

is exactly the ring generated by the colored bubbles, as we proved in Lemma 5.4. The functor $\Sigma_{n,n}$ sends double dots to colored bubbles.

Note also that $S\Pi_{(1^n)} \cong \mathbb{Q}[x_1 - x_2, x_2 - x_3, \ldots, x_{n-1} - x_n]$ and the surjective map $S\Pi_{(1^n)} \to \mathrm{END}_{\mathcal{S}(n,n)}(1_n)$, which we explained in Section 5, is equal to $\Sigma_{n,n}$. This actually shows that $\mathrm{END}_{\mathcal{S}(n,n)}(1_n) \cong S\Pi_{(1^n)}$, which is compatible with our Conjecture 5.5.

For $t > 0$, note that by Corollary 4.11 in [8] $\mathrm{HOM}_{\mathcal{SC}_1(n)}(\emptyset, \underline{\boldsymbol{i}})$ is a free left $\mathrm{HOM}_{\mathcal{SC}_1(n)}(\emptyset, \emptyset)$-module of rank one, generated by the diagram consisting of t StartDots colored $i_1, \ldots, i_t$ respectively. Note also that, by the fullness of $\Psi_{n,n}$ and by Theorem 1.3, Proposition 1.4 and Theorem 2.7 in [16], we know that

$$\mathrm{HOM}_{\mathcal{S}(n,n)}(1_n, \mathcal{E}_{-i_1}\mathcal{E}_{+i_1}\cdots\mathcal{E}_{-i_t}\mathcal{E}_{+i_t}1_n)$$

is a free $\mathrm{END}_{\mathcal{S}(n,n)}(1_n)$-module of rank one generated by the diagram consisting of t cups colored $i_1, \ldots, i_t$ respectively. Our functor $\Sigma_{n,n}$ maps the StartDots to the cups, so we get that

$$\Sigma_{n,n}\colon \mathrm{HOM}_{\mathcal{SC}_1(n)}(\emptyset, \underline{\boldsymbol{i}}) \to \mathrm{HOM}_{\mathcal{S}(n,n)}(1_n, \mathcal{E}_{-i_1}\mathcal{E}_{+i_1}\cdots\mathcal{E}_{-i_t}\mathcal{E}_{+i_t}1_n)$$

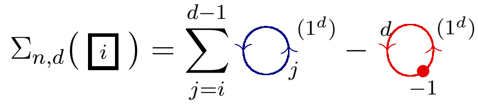
is an isomorphism. $\square$

6.5. **A functor from $\mathcal{SC}_1'(d)$ to $\mathcal{S}(n,d)^*((1^d),(1^d))$ for $d < n$.** Let $d < n$ be arbitrary but fixed. For $(1^d) \in \Lambda(n,d)$, we write $1_d = 1_{(1^d)}$. We define a monoidal additive $\mathbb{Q}$-linear functor

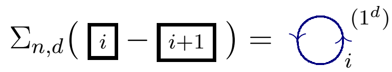
$$\Sigma_{n,d}\colon \mathcal{SC}_1'(d) \to \mathcal{S}(n,d)^*((1^d),(1^d)),$$

which is very similar to $\Sigma_{n,n}$ from the previous subsection and categorifies $\sigma_{n,d}$ of Section 2. Recall that $\mathcal{SC}_1(d) \subseteq \mathcal{SC}_1'(d)$ is a faithful subcategory. So we define $\Sigma_{n,d}$ in exactly the same way as $\Sigma_{n,n}$, but restricting to the colors $1 \le i \le d-1$ and sending $\emptyset$ to the empty diagram in the region labeled

(1^d) instead of (1^n). The only new ingredient for the definition of $\Sigma_{n,d}$ is the image of the boxes, which we define by



for any $i = 1, \ldots, d$. Note that we have



which agrees with the first box relation (6.19). One easily checks that $\Sigma_{n,d}$ preserves the other box relations as well. The rest of the proof that $\Sigma_{n,d}$ is well-defined uses the same arguments as in the previous subsection.

As in Subsection 6.3 we have

Lemma 6.8. There is a commutative diagram

$$\begin{tikzcd}
\mathcal{SC}'_1(d) \arrow[rr, "\mathcal{F}'_{EK}"] \arrow[dr, "\Sigma_{n,d}"'] & & \mathbf{Bim}(d)^* \\
& \mathcal{S}(n,d)^*((1^d),(1^d)) \arrow[ur, "\mathcal{F}_{Bim}"'] &
\end{tikzcd}$$

Proposition 6.9. The functor $\Sigma_{n,d}$ is an equivalence of categories.

Proof. Note that $\mathcal{E}_{+k}1_d = 0$, for any $k \geq d$, so by the commutation isomorphisms in $\mathcal{S}(n,d)$ we see that any object x in $\mathcal{S}(n,d)^*((1^d),(1^d))$ is isomorphic to a direct sum of objects whose factors are all of the form $\mathcal{E}_{-i}\mathcal{E}_{+i}1_d$ with $1 \leq i \leq d-1$. This is a consequence of the commutation relations on the decategorified level [7] which become commutation isomorphisms on the category level. Therefore $\Sigma_{n,d}$ is essentially surjective. Faithfulness follows from Elias and Khovanov's results and the commuting triangle in Lemma 6.8, just as in the previous subsection.

The arguments which show that $\Sigma_{n,d}$ is full are almost identical to the ones in the previous subsection. The only difference is that we now have

$$\mathrm{HOM}_{\mathcal{SC}'_1(d)}(\emptyset,\emptyset) \cong \mathbb{Q}[x_1,\ldots,x_d] \cong \mathrm{END}_{\mathcal{S}(\mathrm{n},\mathrm{d})}(1_d).$$

The first isomorphism follows from Elias and Khovanov's results in [8]. The second isomorphism follows from the fact that the i-colored bubbles of positive degree are all zero for $i > d$, since their inner regions are labeled by elements that do not belong to $\Lambda(n,d)$, and the d-colored bubble with a dot is mapped to x_d. Therefore we have

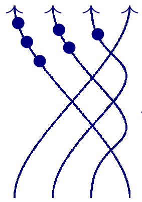
$$\mathrm{END}_{\mathcal{S}(\mathrm{n},\mathrm{d})}(1_d) \cong \mathbb{Q}[x_1 - x_2, \ldots, x_{d-1} - x_d, x_d] \cong \mathbb{Q}[x_1,\ldots,x_d].$$

□

7. Grothendieck algebras

7.1. **The Grothendieck algebra of $\mathcal{S}(n,d)$.** To begin with, let us introduce some notions and notations analogous to Khovanov and Lauda's in Section 3.5 in [16]. Let $\dot{\mathcal{U}}(\mathfrak{sl}_n)$ and $\dot{\mathcal{S}}(n,d)$ denote the Karoubi envelopes of $\mathcal{U}(\mathfrak{sl}_n)$ and $\mathcal{S}(n,d)$ respectively. We define the objects of $\dot{\mathcal{S}}(n,d)$ to be the elements in $\Lambda(n,d)$ and we define the hom-category $\dot{\mathcal{S}}(n,d)(\lambda,\mu)$ to be the usual Karoubi

envelope of $\mathcal{S}(n,d)(\lambda,\mu)$, for any $\lambda,\mu\in\Lambda(n,d)$. There exist idempotents $e\in\mathrm{End}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\mathbf{i}}1_\lambda)$, so that $(\mathcal{E}_{\mathbf{i}},e)$ is a direct summand of $\mathcal{E}_{\mathbf{i}}$ in $\dot{\mathcal{S}}(n,d)$. For example, we can define the idempotents

$e_{+i,m,\lambda}=$  λ, $\qquad e_{-i,m,\lambda}=(-1)^{\frac{m(m-1)}{2}}$  λ

in $\mathrm{End}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{+i^m}1_\lambda)$ and $\mathrm{End}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{-i^m}1_\lambda)$ respectively. We can define the 1-morphisms in $\dot{\mathcal{S}}(n,d)$

$$\mathcal{E}_{\pm i^{(m)}}1_\lambda:=(\mathcal{E}_{\pm i^m}1_\lambda,e_{\pm i,m,\lambda})\left\{\frac{m(1-m)}{2}\right\}$$

and have

$$\mathcal{E}_{\pm i^m}1_\lambda\cong(\mathcal{E}_{\pm i^{(m)}}1_\lambda)^{\oplus[m]!}.$$

Recall that $[m]!\in\mathbb{N}[q,q^{-1}]$ is the q-factorial $[m][m-1]\cdots 1$, with $[s]=(q^s-q^{-s})/(q-q^{-1})$. For any q-integer $\oplus_{n=-j}^k a_nq^n\in\mathbb{N}[q,q^{-1}]$, we define

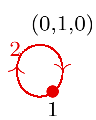
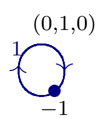
$$A^{\oplus_{n=-j}^k a_nq^n}=\bigoplus_{n=-j}^k\left(\oplus_{i=1}^{a_n}A\{n\}\right).$$

Note that $e_{+i,m,\lambda}=0$ for $m>\lambda_{i+1}$ and $e_{-i,m,\lambda}=0$ for $m>\lambda_i$, because for those values of m the left-most region of their defining diagrams has a label with a negative entry. This shows that these idempotents depend on λ, which was not the case in [16]. Note that these lower bounds for m are sharp, i.e.

$$\mathcal{E}_{+i^{(m)}}1_\lambda=0\Leftrightarrow m>\lambda_{i+1}$$
$$\mathcal{E}_{-i^{(m)}}1_\lambda=0\Leftrightarrow m>\lambda_i.$$

This follows from observing the image of $\mathcal{E}_{\pm i^{(m)}}1_\lambda$ under the 2-functor $\mathcal{F}_{Bim}\colon\mathcal{S}(n,d)^*\to\mathbf{Bim}^*$.

Before we go on, let us make the remark alluded to above Conjecture 5.5, when we showed that

(7.1)  $-$  $=\ 0.$

Remark 7.1. Suppose $\lambda=(\ldots,a,0,\ldots)\in\Lambda(n,d)$, with a in the ith position. Let $\mu=(\ldots,0,a,\ldots)$ be obtained from λ by switching a and 0. From Theorem 5.6 and Corollary 5.8 in [18] it follows that

$$\mathcal{E}_{i^{(a)}}\mathcal{E}_{-i^{(a)}}1_\lambda\cong 1_\lambda\quad\text{and}\quad\mathcal{E}_{-i^{(a)}}\mathcal{E}_{i^{(a)}}1_\mu\cong 1_\mu,$$

because we have $\mathcal{E}_{i^{(j)}}1_\lambda=0$ and $\mathcal{E}_{-i^{(j)}}1_\mu=0$ in $\dot{\mathcal{S}}(n,d)$ for any $j>0$. Therefore λ and μ are isomorphic objects in the 2-category $\dot{\mathcal{S}}(n,d)$. Our proof of (7.1) used the 2-isomorphism between $1_{(0,1,0)}$ and $\mathcal{E}_{-1}\mathcal{E}_1 1_{(0,1,0)}$ explicitly in the first step.

Note that $\dot{\mathcal{S}}(n,d)$ is Krull-Schmidt, just as $\dot{\mathcal{U}}(\mathfrak{sl}_n)$. Therefore, we can take the split Grothendieck algebras/categories $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{U}}(\mathfrak{sl}_n))$ and $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$. Considering the latter as a category, we follow Khovanov and Lauda [16] and define $\Lambda(n,d)$ to be the set of objects. The hom-space $\hom(\lambda,\mu)$ we define to be the split Grothendieck algebra of the additive category $\dot{\mathcal{S}}(\lambda,\mu)$. Alternatively, we can see this as an (idempotented) algebra rather than a category. In the sequel we will

use both points of view interchangeably. Note that the remark above shows that there are objects in $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$ which are isomorphic, e.g. $(1,0,0),(0,1,0)$ and $(0,0,1)$ are all isomorphic in $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(3,1))$.

Analogous to Khovanov and Lauda's homomorphism $\gamma = \gamma_U\colon \dot{\mathbf{U}}(\mathfrak{sl}_n) \to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{U}}(\mathfrak{sl}_n))$, we define a homomorphism $\gamma_S\colon \dot{\mathbf{S}}(n,d) \to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$ by

$$E_{s_1}\cdots E_{s_m}1_\lambda \mapsto [\mathcal{E}_{s_1}\cdots\mathcal{E}_{s_m}1_\lambda].$$

Our main goal in this section is to prove that γ_S is an isomorphism. Recall that in order to show that γ_U is an isomorphism, Khovanov and Lauda had to determine the indecomposable direct summands of certain 1-morphisms x in $\dot{\mathcal{U}}(\mathfrak{sl}_n)$. They did this by looking at $K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(x))$, which is the Grothendieck group of the finitely generated graded projective $\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(x)$-modules. This allowed them to use known results about the Grothendieck groups of graded algebras, which we recall below. The connection between the two sorts of Grothendieck groups relies on the fact that a finitely-generated graded projective $\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(x)$-module is determined by an idempotent e in $\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(x)$ and $[(x,e)]$ is an element of $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{U}}(\mathfrak{sl}_n))$. The isomorphism classes of indecomposable projective modules form a basis of $K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(x))$ and correspond to the minimal idempotents in $\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(x)$. We refer to [16] for more details. We will follow Khovanov and Lauda's approach closely to show that γ_S is surjective, but will use a completely different method to show that γ_S is injective. Although we have tried to explain our results clearly, we suspect that the part of this section which deals with the surjectivity of γ_S will be quite hard to understand for someone unfamiliar with [14, 15, 16, 20]. The part on the injectivity of γ_S can probably be read independently.

Before we move on to our results in this section, we should recall the basic facts about Grothendieck groups of (graded) algebras which Khovanov and Lauda explained in Subsections 3.8.1 and 3.8.2 in [16]. If A is a finite-dimensional algebra over a field, let $K_0(A)$ be the Grothendieck group of the category of the finitely generated projective A-modules.

Proposition 7.2. Let $f\colon A \to B$ be a surjective homomorphism between two finite-dimensional algebras, then $K_0(f)\colon K_0(A) \to K_0(B)$ is surjective.

Unfortunately in the applications in [16] and in our paper, the algebras involved are not finite-dimensional. But fortunately they are $\mathbb{Z}$-graded and we can resort to finite-dimensional quotients which do not alter the Grothendieck groups. Let A be a $\mathbb{Z}$-graded algebra over a field, such that in each degree it has finite dimension and the grading is bounded from below.

Definition 7.3. Let $I \subset A$ be a two-sided homogeneous ideal. We say that I is *virtually nilpotent* if for each degree $a \in \mathbb{Z}$ there exists an $N > 0$ such that the degree a summand of I^N is equal to zero.

Lemma 7.4. Let $I \subset A$ be a virtually nilpotent ideal. Then $K_0(A) \cong K_0(A/I)$.

Corollary 7.5. Let $f\colon A \to B$ be a degree preserving homomorphism of $\mathbb{Z}$-graded algebras of the type described above, and $I \subset A$ a virtually nilpotent ideal of finite codimension. If f is surjective, then $K_0(f)\colon K_0(A) \cong K_0(A/I) \to K_0(B/f(I)) \cong K_0(B)$ is surjective.

We also need a fact about the split Grothendieck group of Krull-Schmidt categories. This result is not recalled in [16], but is well known in homological algebra. We thank Mikhail Khovanov for explaining it to us. To help the reader, we briefly sketch the proof below.

Proposition 7.6. Let $\mathcal{F}\colon C \to D$ be an additive $\mathbb{Q}$-linear degree preserving functor between two graded Krull-Schmidt categories, whose hom-spaces are finite-dimensional in each degree and whose gradings are bounded from below. If $\mathcal{F}$ is fully faithful, then $K_0(\mathcal{F})\colon K_0(C) \to K_0(D)$ is injective.

Since C and D are Krull-Schmidt, each object in C or D can be uniquely decomposed into indecomposables, which generate $K_0(C)$ and $K_0(D)$ respectively. Being fully faithful, $\mathcal{F}$ maps the set of indecomposables in C injectively into the set of indecomposables in D.

We now get to the main part of this section. By simply checking the definitions, we see that the following square commutes:

$$\begin{array}{ccc} \dot{\mathbf{U}}(\mathfrak{sl}_n) & \xrightarrow{\gamma_U} & K_0^{\mathbb{Q}(q)}(\dot{\mathcal{U}}(\mathfrak{sl}_n)) \\ {\scriptstyle\varphi_{n,d}}\downarrow & & \downarrow{\scriptstyle K_0^{\mathbb{Q}(q)}(\Psi_{n,d})} \\ \dot{\mathbf{S}}(n,d) & \xrightarrow{\gamma_S} & K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)). \end{array} \tag{7.2}$$

We know that $\varphi_{n,d}$ is surjective and γ_U is an isomorphism. We also know that $\Psi_{n,d}$ is full, but we cannot automatically conclude that $K_0^{\mathbb{Q}(q)}(\Psi_{n,d})$ is surjective, because $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(x)$ is infinite-dimensional for any 1-morphism x. We want to prove that $K_0^{\mathbb{Q}(q)}(\Psi_{n,d})$ and γ_S are surjective. Of course it suffices to prove that γ_S is surjective.

Let us first sketch the chain of arguments that leads to the proof of the surjectivity of γ_U in Theorem 1.1 in [16]. The proof is by induction with respect to the width of an indecomposable 1-morphism P in $\mathcal{U}(\mathfrak{s}l_n)$, which by definition is the smallest non-negative integer m such that P is isomorphic to a direct summand of $\mathcal{E}_{\mathbf{i}}1_\lambda\{t\}$ with $||\mathbf{i}|| = m$. In Lemma 3.38 Khovanov and Lauda prove that any indecomposable object of width m is isomorphic to a direct summand of $\mathcal{E}_{\nu,-\nu'}1_\lambda\{t\}$, for certain $\lambda \in \mathbb{Z}^{n-1}$, $t \in \mathbb{Z}$ and $\nu,\nu' \in \mathbb{N}[I]$, such that $||\nu|| + ||\nu'|| = m$. This narrows down the number of cases that need to be considered in the proof of Theorem 1.1.

Next, suppose P has width zero, then $P \cong 1_\lambda$ up to a shift, and $K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_\mathrm{n})}(1_\lambda))$ lies in the image of γ_U, because it is isomorphic to $\mathbb{Q}$ with generator $[1_\lambda]$. The induction step relies on the exact sequence of rings (3.38)

$$0 \to I_{\nu,-\nu',\lambda} \to \mathrm{END}_{\mathcal{U}(\mathfrak{sl}_\mathrm{n})}(\mathcal{E}_{\nu,-\nu'}1_\lambda) \to R_{\nu,-\nu',\lambda} \to 0. \tag{7.3}$$

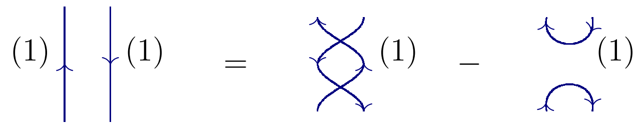
Recall that for $\mathfrak{g} = \mathfrak{sl}_n$, the ring $R_{\nu,-\nu',\lambda}$ is isomorphic to that of 2-morphisms whose diagrams are split into upward strands with source and target belonging to ν, downward strands with source and target belonging to $-\nu'$, and bubbles on the right-hand side. The ideal $I_{\nu,-\nu',\lambda}$ is generated by diagrams which contain at least one cup or cap between ν and $-\nu'$. Note that the latter are precisely the 2-morphisms which factor through a direct sum of objects with width smaller than $||\nu|| + ||\nu'||$. As they remark in Remark 3.18, this exact sequence is split for $\mathfrak{g} = \mathfrak{sl}_n$. Therefore there is a direct sum decomposition

$$K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_\mathrm{n})}(\mathcal{E}_{\nu,-\nu'}1_\lambda)) \cong K_0^{\mathbb{Q}(q)}(I_{\nu,-\nu',\lambda}) \oplus K_0^{\mathbb{Q}(q)}(R_{\nu,-\nu',\lambda}). \tag{7.4}$$

The fact that $K_0^{\mathbb{Q}(q)}(R_{\nu,-\nu',\lambda})$ lies in the image of γ_U is essentially a consequence of the results in [14, 15] and a technical result involving a virtually nilpotent ideal, the details of which we do not need here. On the other hand, The 2-morphisms in $I_{\nu,-\nu',\lambda}$ factor through direct sums of objects of smaller width, so any minimal idempotent in this ideal corresponds to an object of smaller width. Therefore $K_0^{\mathbb{Q}(q)}(I_{\nu,-\nu',\lambda})$ lies in the image of γ_U by induction. This shows that

$K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(\mathcal{E}_{\nu,-\nu'}1_\lambda))$ lies in the image of γ_U, as had to be proved. We should warn the reader that, contrary to what might seem at a first reading, the direct sum decomposition in (7.4) does not preserve indecomposability. For example, consider the direct sum $EF1_1 \cong FE1_1 \oplus 1_1$ for $n = 2$. This corresponds to the diagrammatic equation

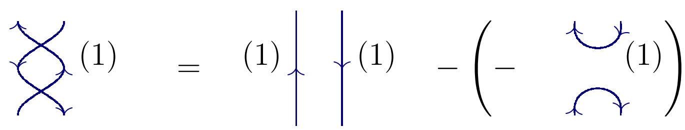
(7.5)



The identity on $EF1_1$ is an indecomposable idempotent in $R_{+,-,(1)}$, but can be decomposed in $\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(\mathcal{E}_{+,-}1_1)$ into the two indecomposable idempotents on the right-hand side of (7.5), which have width 2 and 0 respectively. Note that the second term on the right-hand side belongs to $I_{+,-,(1)}$. So Khovanov and Lauda's homomorphism

$$\beta\colon \mathrm{END}_{\mathcal{U}(\mathfrak{sl}_n)}(\mathcal{E}_{+,-}1_1) \to R_{+,-,(1)}$$

maps the first term on the right-hand side to the identity on $EF1_1$. The map backwards, which they call α, is simply the inclusion, so it maps the identity to the identity. In the induction step above, one therefore writes



to prove that the class of the indecomposable summand of $EF1_1$ of width 2 corresponding to the idempotent on the left-hand side, belongs to the image of γ_U.

Next let us see how Khovanov and Lauda's proofs can be adapted to our setting. In the first place, note that all results in Section 3.5 of [16] continue to be true. More precisely, the statements in their Propositions 3.24, 3.25 and 3.26 are still true, although some direct summands might now be zero depending on the labels of the regions in the diagrams. The crucial Lemma 3.38 in Section 3.8 in [16] holds literally true in our case just as well.

Let us now prove the analogue of their Theorem 1.1. Our proof is essentially the same, except that we use the fact that γ_U is an isomorphism and $\Psi_{n,d}$ is full to avoid having to formulate and use analogues of the results in [14] and [15], which might be hard. This is the reason why we did not go into the details of those results above.

Lemma 7.7. The homomorphism

$$\gamma_S\colon \dot{\mathbf{S}}(n,d) \to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$$

is surjective.

Proof. For the basis of the induction, recall our surjection $S\Pi_\lambda \to \mathrm{END}_{\mathcal{S}(n,d)}(1_\lambda)$ explained in Section 5. The ideal of elements of positive degree $S\Pi_\lambda^+$ is virtually nilpotent of codimension one, so by Corollary 7.5 it follows that

$$\mathbb{Q} \cong K_0^{\mathbb{Q}(q)}(S\Pi_\lambda) \to K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{S}(n,d)}(1_\lambda))$$

is surjective. Therefore $K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{S}(n,d)}(1_\lambda))$ is generated by $[1_\lambda]$, i.e. 1_λ is also indecomposable in our case. Since $\gamma_S(1_\lambda) = [1_\lambda]$, we see that $K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{S}(n,d)}(1_\lambda))$ lies in the image of γ_S. Note that we have not yet proved that $[1_\lambda] \neq 0$. After we have proved that $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)) \cong \dot{\mathbf{S}}(n,d)$ in Theorem 7.11, it follows that $K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{S}(n,d)}(1_\lambda)) \cong \mathbb{Q}$ with $[1_\lambda] \neq 0$ being the generator.

For the induction step, note that $\Psi_{n,d}$ maps the exact sequence (7.3) surjectively onto the exact sequence

$$0 \to \Psi_{n,d}(I_{\nu,-\nu',\overline{\lambda}}) \to \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\nu,-\nu'}1_{\lambda}) \to \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\nu,-\nu'}1_{\lambda})/\Psi_{n,d}(I_{\nu,-\nu',\overline{\lambda}}) \to 0. \tag{7.6}$$

We do not know if this exact sequence is split, but fortunately it does not matter for our purpose.

Note also that $\Psi_{n,d}$ induces a surjective map

$$R_{\nu,-\nu',\lambda} \to \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\nu,-\nu'}1_{\lambda})/\Psi_{n,d}(I_{\nu,-\nu',\overline{\lambda}}).$$

Recall that Khovanov and Lauda defined a virtually nilpotent ideal $\beta\alpha(J) \subset R_{\nu,-\nu',\overline{\lambda}}$ of codimension one in Section 3.8.3 in [16], alluded to above. By Corollary 7.5 this implies that

$$K_0^{\mathbb{Q}(q)}(R_{\nu,-\nu',\overline{\lambda}}) \to K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\nu,-\nu'}1_{\lambda})/\Psi_{n,d}(I_{\nu,-\nu',\overline{\lambda}})) \tag{7.7}$$

is surjective. Now, just as in the proof of Theorem 1.1, let $e \in \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\nu,-\nu'}1_{\lambda})$ be a minimal idempotent of width m, with $||\nu|| + ||\nu'|| = m$. We have to show that $[(\mathcal{E}_{\nu,-\nu'}1_{\lambda}, e)]$ lies in the image of γ_S. Let $\overline{e}$ be the image of e in $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\nu,-\nu'}1_{\lambda})/\Psi_{n,d}(I_{\nu,-\nu',\overline{\lambda}})$. Note that we do not know a priori that $\overline{e}$ is indecomposable, but that does not matter. By the surjectivity of (7.7), we can lift $\overline{e}$ to an idempotent $e' \in R_{\nu,-\nu',\overline{\lambda}}$. By Khovanov and Lauda's results, we know that

$$[(\mathcal{E}_{\nu,-\nu'}1_{\overline{\lambda}}, e')] \in K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{U}(\mathfrak{sl}_\mathrm{n})}(\mathcal{E}_{\nu,-\nu'}1_{\lambda})) \subseteq K_0^{\mathbb{Q}(q)}(\dot{\mathcal{U}}(\mathfrak{sl}_n))$$

is in the image of γ_U. By the commutativity of the square in (7.2), this implies that

$$[(\mathcal{E}_{\nu,-\nu'}1_{\lambda}, \Psi_{n,d}(e'))] \in K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\nu,-\nu'}1_{\lambda})) \subseteq K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$$

is in the image of γ_S. Note that $e - \Psi_{n,d}(e')$ maps to zero in $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(\mathcal{E}_{\nu,-\nu'}1_{\lambda})/\Psi_{n,d}(I_{\nu,-\nu',\overline{\lambda}})$. By the minimality of e, we therefore have $\Psi_{n,d}(e') = e + e''$, with e'' an orthogonal idempotent in $\Psi_{n,d}(I_{\nu,-\nu',\overline{\lambda}})$ which can be decomposed into minimal idempotents of width $< m$. By induction $[(\mathcal{E}_{\nu,-\nu'}1_{\lambda}, e'')]$ is contained in the image of γ_S. This shows that $[(\mathcal{E}_{\nu,-\nu'}1_{\lambda}, e)]$ is contained in the image of γ_S too, as we had to show. $\square$

The following two corollaries are immediate.

Corollary 7.8. The homomorphism

$$K_0^{\mathbb{Q}(q)}(\Psi_{n,d})\colon K_0^{\mathbb{Q}(q)}(\dot{\mathcal{U}}(\mathfrak{sl}(n))) \to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$$

is surjective.

Corollary 7.9. $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$ is a quotient of $\dot{\mathbf{S}}(n,d)$. In particular $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$ is finite-dimensional and semi-simple.

Before we prove the main result of this paper, we first categorify the homomorphism $\iota_{n,m}$ from Section 2. Let $m \geq n$ and d arbitrary. Let $\Xi_{n,m} = \oplus_{\lambda\in\Lambda(n,d)}1_{\lambda} \in \mathcal{S}(m,d)$. Let $\mathcal{S}(n,m,d)$ be the full sub-2-category of $\mathcal{S}(m,d)$ whose objects belong to $\Lambda(n,d) \subseteq \Lambda(m,d)$.

Definition 7.10. Let $m \geq n$ and d arbitrary. We define a functor

$$\mathcal{I}_{n,m}\colon \mathcal{S}(n,d) \to \mathcal{S}(n,m,d)$$

by mapping any diagram in $\mathcal{S}(n,d)$ to itself, using the inclusion $\Lambda(n,d) \subseteq \Lambda(m,d)$ for the labels.

It is easy to see that $\mathcal{I}_{n,m}$ is well-defined and essentially surjective. We conjecture it to be faithful, but have no proof. It is certainly not full, because $\mathcal{S}(n,m,d)$ contains n-colored bubbles for example. Perhaps there is a virtually nilpotent ideal $I \subset \mathcal{S}(n,m,d)$ such that $\mathcal{S}(n,d) \cong \mathcal{S}(n,m,d)/I$, e.g. the ideal generated by all diagrams with n-colored bubbles of positive degree on the right-hand side.

Theorem 7.11. The homomorphism

$$\gamma_S \colon \dot{\mathbf{S}}(n,d) \to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$$

is an isomorphism.

Proof. After the result of Lemma 7.7 it only remains to show that $K_0^{\mathbb{Q}(q)}(\mathcal{S}(n,d))$ and $\dot{\mathbf{S}}(n,d)$ have the same dimension.

We first show the case $n = d$. Let $1_n = 1_{(1^n)}$. In Proposition 6.7 we proved that $\mathcal{SC}_1(n) \cong \mathcal{S}(n,n)^*((1^n),(1^n))$ is a full sub-2-category of $\mathcal{S}(n,n)^*$. By Proposition 7.6 this implies

$$K_0^{\mathbb{Q}(q)}(\mathcal{SC}(n)) \cong K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,n)((1^n),(1^n))) \subseteq K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,n)).$$

By Theorem 6.4 we know that $K_0^{\mathbb{Q}(q)}(\mathcal{SC}(n))$ is isomorphic to $H_q(n)$. Thus Lemma 2.13 implies that $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,n)) \cong \dot{\mathbf{S}}(n,n)$.

Now let $d < n$. In Proposition 6.9 we proved that $\mathcal{SC}'_1(d) \cong \mathcal{S}(n,d)^*((1^d),(1^d))$ is a full sub-2-category of $\mathcal{S}(n,d)^*$. By Proposition 7.6 this implies

$$K_0^{\mathbb{Q}(q)}(\mathcal{SC}'(d)) \cong K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)((1^d),(1^d))) \subseteq K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)).$$

By Theorem 6.4 we know that $K_0^{\mathbb{Q}(q)}(\mathcal{SC}'(d))$ is isomorphic to $H_q(d)$. Thus Lemma 2.13 shows that $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)) \cong \dot{\mathbf{S}}(n,d)$.

Next, assume that $n < d$. Consider the functor

$$\mathcal{I}_{n,d} \colon \mathcal{S}(n,d) \to \mathcal{S}(n,d,d).$$

We have the following commuting square

$$\begin{tikzcd}
\dot{\mathbf{S}}(n,d) \arrow[r, "\iota_{n,d}"] \arrow[d, "\gamma_S(n)"'] & \xi_{n,d}\dot{\mathbf{S}}(d,d)\xi_{n,d} \arrow[d, "\gamma_S(d)"] \\
K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)) \arrow[r, "K_0^{\mathbb{Q}(q)}(\mathcal{I}_{n,d})"] & K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d,d)).
\end{tikzcd}$$

We already know that $\gamma_S(d) \colon \dot{\mathbf{S}}(d,d) \to K_0^{\mathbb{Q}(q)}(\mathcal{S})(d,d))$ is an isomorphism from the first case we proved. Therefore $\gamma_S(d) \colon \xi_{n,d}\dot{\mathbf{S}}(n,d)\xi_{n,d} \to [\Xi_{n,d}]K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(d,d))[\Xi_{n,d}] \cong K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d,d))$ is an isomorphism as well. Recall that $\iota_{n,d}$ is an isomorphism. It follows that $\gamma_S(n)$ is injective. Recall that $\gamma_S(n)$ is surjective, by Lemma 7.7. It follows that $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)) \cong \dot{\mathbf{S}}(n,d)$. $\square$

Note that we did not follow Khovanov and Lauda's approach to prove injectivity of γ_S. Recall that they defined a non-degenerate $\mathbb{Q}$-semilinear form on $\dot{U}(\mathfrak{sl}_n)$, which is closely related to Lusztig's bilinear form in [22], and defined an inner product on $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{U}}(\mathfrak{sl}_n))$ by

$$\langle [x],[y] \rangle = \dim_q(\mathrm{HOM}_{\dot{\mathcal{U}}(\mathfrak{sl}_n)}(x,y)).$$

They showed that γ_U is injective by proving that it is an isometry. We could not prove that γ_S is injective in this way, because we could not find such a $\mathbb{Q}$-semilinear form on $\dot{\mathbf{S}}(n,d)$ in the

literature.[4] By our Theorem 7.11, we can define one now. We first define a non-degenerate $\mathbb{Q}$-semilinear form on $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$ as above

$$\langle [x],[y]\rangle = \dim_q(\mathrm{HOM}_{\dot{\mathcal{S}}(\mathrm{n,d})}(x,y)).$$

Definition 7.12. We define a non-degenerate $\mathbb{Q}$-semilinear form on $\dot{\mathbf{S}}(n,d)$ by

$$\langle x,y\rangle = \langle \gamma_S(x),\gamma_S(y)\rangle.$$

By definition γ_S is an isometry. It is easy to see that the semilinear form on $\dot{\mathbf{S}}(n,d)$ has the following properties (compare to Proposition 2.4 in [16]):

Corollary 7.13. We have

(1) $\langle 1_{\lambda_1}x1_{\lambda_2}, 1_{\lambda_1'}x1_{\lambda_2'}\rangle = 0$ for all x,y unless $\lambda_1=\lambda_1'$ and $\lambda_2=\lambda_2'$.
(2) $\langle ux,y\rangle = \langle x,\tau(u)y\rangle$.

However, Khovanov and Lauda's interpretation of the semilinear form on $\dot{U}(\mathfrak{sl}_n)$ in Theorem 2.7 in [16], which shows that $\langle [x],[y]\rangle = \dim_q(\mathrm{HOM}_{\dot{\mathcal{U}}(\mathfrak{sl}_\mathrm{n})}(x,y))$ can be obtained by counting the number of minimal diagrams in each degree in $\mathrm{HOM}_{\dot{\mathcal{U}}(\mathfrak{sl}_\mathrm{n})}(x,y)$, does not hold in our case. This is because minimal diagrams in $\mathcal{S}(n,d)$ are not linearly independent in general. For example, consider relation (3.11) for $n=2$ and $\lambda=(1,0)$. Note that the sum on the right-hand side only contains one term. The first term on the right-hand side, i.e. the one with the two crossings, has a middle region with label $(2,-1)\notin\Lambda(2,1)$, so it is equal to zero. This shows that the minimal diagram on the left-hand side is equivalent to the minimal diagram on the right-hand side.

7.2. **Categorical Weyl modules.** We conjecture that it is easy to categorify the irreducible representations V_λ, for $\lambda\in\Lambda^+(n,d)$, using the category $\mathcal{S}(n,d)$. Recall from Lemma 2.10 that

$$V_\lambda \cong \dot{\mathbf{S}}(n,d)1_\lambda/[\mu>\lambda].$$

Definition 7.14. For any $\lambda\in\Lambda^+(n,d)$, let $\mathcal{S}(n,d)1_\lambda$ be the category whose objects are the 1-morphisms in $\mathcal{S}(n,d)$ of the form $x1_\lambda$ and whose morphisms are the 2-morphisms in $\mathcal{S}(n,d)$ between such 1-morphisms. Note that $\mathcal{S}(n,d)1_\lambda$ does not have a monoidal structure, because two 1-morphisms $x1_\lambda$ and $y1_\lambda$ cannot be composed in general. Alternatively one can see $\mathcal{S}(n,d)1_\lambda$ as a graded ring, whose elements are the morphisms.

Let $\mathcal{V}_\lambda$ be the quotient of $\mathcal{S}(n,d)1_\lambda$ by the ideal generated by all diagrams which contain a region labeled by $\mu>\lambda$.

Note that there is a natural categorical action of $\mathcal{S}(n,d)$, and therefore of $\mathcal{U}(\mathfrak{s}l_n)$, on $\mathcal{V}_\lambda$, defined by putting a diagram in $\mathcal{S}(n,d)$ on the left-hand side of a diagram in $\mathcal{V}_\lambda$. This action descends to an action of $\dot{\mathbf{S}}(n,d)\cong K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d))$ on $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda)$, where $\dot{\mathcal{V}}_\lambda$ is the Karoubi envelope of $\mathcal{V}_\lambda$. Note that γ_S induces a well-defined linear map $\gamma_\lambda\colon V_\lambda\to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda)$, which intertwines the $\dot{\mathbf{S}}(n,d)$-actions.

Lemma 7.15. The linear map γ_λ is surjective.

[4] Williamson defines such a form in [39] for $n=d$, but we do not know of any diagrammatic interpretation of his form even in that restricted case. We conjecture that his form is equivalent to ours for $n=d$. This is the only related form in the literature that we could find, even after asking numerous experts.

Proof. We first show that $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)1_\lambda) \to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda)$ is surjective. Again, we want to use Proposition 7.2, but have to be careful because the graded rings involved are not finite-dimensional. Choose an object $x \in \mathcal{S}(n,d)1_\lambda$. Recall that $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(x)$ is finitely generated as a right module over $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_\lambda)$. Let $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_\lambda)^+ \subseteq \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_\lambda)$ be the two-sided ideal of 2-morphisms of strictly positive degree. Note that $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_\lambda)^+$ is a codimension one virtually nilpotent ideal. Let $\mathrm{END}^+_{\mathcal{S}(n,d)}(x) \subseteq \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(x)$ be the image of $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(x) \otimes \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_\lambda)^+$ under the right action. Then $\mathrm{END}^+_{\mathcal{S}(n,d)}(x)$ is a two-sided ideal of finite codimension and is virtually nilpotent, because the grading of $\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(1_\lambda)$ is bounded from below.

Now let $\mathrm{END}^{>\lambda}_{\mathcal{S}(n,d)}(x) \subseteq \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(x)$ be the two-sided ideal generated by all diagrams with at least one region labeled by a $\mu > \lambda$. By Corollary 7.5, the projection

$$\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(x) \to \mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(x)/\mathrm{END}^{>\lambda}_{\mathcal{S}(n,d)}(x)$$

induces a surjective homomorphism

$$K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(x)) \to K_0^{\mathbb{Q}(q)}(\mathrm{END}_{\mathcal{S}(\mathrm{n,d})}(x)/\mathrm{END}^{>\lambda}_{\mathcal{S}(n,d)}(x)).$$

Since x was arbitrary, it follows that

$$K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)1_\lambda) \to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda)$$

is surjective. Thus, the composite linear map

$$\dot{\mathbf{S}}(n,d)1_\lambda \cong K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)1_\lambda) \to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda)$$

is surjective. Note that $[\mu > \lambda]$ is contained in the kernel of this map, which proves this lemma. $\square$

Conjecture 7.16. For any $\lambda \in \Lambda^+(n,d)$, we have

$$K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda) \cong V_\lambda.$$

We do not know how to prove the conjecture in general. Note that by Lemma 7.15, we have a surjective linear map $\gamma_\lambda \colon V_\lambda \to K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda)$, which intertwines the $\dot{\mathbf{S}}(n,d)$-actions. Since V_λ is irreducible, we have $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda) \cong V_\lambda$ or $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda) = 0$. So it suffices to show that $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_\lambda) \neq 0$. Particular cases can be proved easily. For example, if $\lambda = (d)$, then $\mathcal{V}_\lambda = \mathcal{S}(n,d)1_\lambda$, because there are no weights higher than (d). By Theorem 7.11 we have $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{S}}(n,d)1_\lambda) \cong \dot{\mathbf{S}}(n,d)1_\lambda$, which proves the conjecture in this case.

We can also prove the case $n = 2$. If $\lambda = (d,0)$, then the result follows from the previous case. Suppose $\lambda = (d-c,c)$, for $0 < 2c \leq d$. Note that $(d-2c,0) = (d-c,c)-(c,c) \in \Lambda^+(2,d-2c)$. Recall that we have a functor

$$\Pi_{d,d-2c} \colon \mathcal{S}(2,d) \to \mathcal{S}(2,d-2c),$$

which induces a functor

$$\Pi_{d,d-2c} \colon \mathcal{V}_{(d-c,c)} \to \mathcal{V}_{(d-2c,0)}.$$

Thus we have the following commuting square:

$$\begin{tikzcd}
V_{(d-c,c)} \arrow[r, "\pi_{d,d-2c}"] \arrow[d, "\gamma_{(d-c,c)}"'] & V_{(d-2c,0)} \arrow[d, "\gamma_{(d-2c,0)}"] \\
K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_{(d-c,c)}) \arrow[r, "K_0^{\mathbb{Q}(q)}(\Pi_{d,d-2c})"] & K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_{(d-2c,0)})
\end{tikzcd}$$

We know that $\pi_{d,d-2c}$ and $\gamma_{(d-2c,0)}$ are isomorphisms and $\gamma_{(d-c,c)}$ is surjective. Therefore $\gamma_{(d-c,c)}$ is an isomorphism too, so $K_0^{\mathbb{Q}(q)}(\dot{\mathcal{V}}_{(d-c,c)}) \cong V_{(d-c,c)}$.

There is an obvious functor from the Khovanov-Lauda [14] cyclotomic quotient category $R(*, \lambda)$ to a quotient of our $\mathcal{V}_\lambda$. The quotient is obtained by putting all bubbles of positive degree in the right-most region of the diagrams, labeled λ, equal to zero. By our observations above about $\mathrm{END}^+_{\mathcal{S}(n,d)}(x)$, this quotient has the same Grothendieck group as $\mathcal{V}_\lambda$. The functor is the "identity" on objects and morphisms. The reduction to bubbles argument before Conjecture 5.6 shows that our quotient satisfies the cyclotomic condition. The functor is clearly essentially surjective and full and we conjecture it to be faithful, so that it would be an equivalence of categories.

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DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DO ALGARVE, CAMPUS DE GAMBELAS, 8005-139 FARO, PORTUGAL AND CAMGSD, INSTITUTO SUPERIOR TÉCNICO, AVENIDA ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL
E-mail address: mmackaay@ualg.pt

INSTITUTO DE SISTEMAS E ROBÓTICA AND CAMGSD, INSTITUTO SUPERIOR TÉCNICO, AVENIDA ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL AND, MATHEMATICAL INSTITUTE SANU, KNEZ MIHAILOVA 36, 11000 BEOGRAD, SERBIA
E-mail address: mstosic@math.ist.utl.pt

CAMGSD, INSTITUTO SUPERIOR TÉCNICO, AVENIDA ROVISCO PAIS, 1049-001 LISBOA, PORTUGAL
E-mail address: pedro.forte.vaz@ist.utl.pt